The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2013-14**

Fields **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) For each of the subsets J_1, J_2 of $\mathbb{C}$ specified below determine, with justification, whether it is a subfield of $\mathbb{C}$:

(a) $J_1 = \{a + bi\sqrt{7} : \ a, b \in \mathbb{Q}\}$, ***(5 marks)***

(b) $J_2 = \{a + b\sqrt{3} + ci + di\sqrt{-3} : \ a, b, c, d \in \mathbb{Q}\}$. ***(4 marks)***

(ii) (a) Explain what is meant by saying that a field K is a *finite field extension* of $\mathbb{Q}$. ***(2 marks)***

(b) Let K and L be finite field extensions of $\mathbb{Q}$ such that $K, L \subseteq \mathbb{C}$, and M be the subfield of $\mathbb{C}$ generated by K and L. Show that M is a finite field extension of $\mathbb{Q}$ and that

$$[M : \mathbb{Q}] \leq [K : \mathbb{Q}][L : \mathbb{Q}].$$ ***(9 marks)***

(c) Suppose that $K = \mathbb{Q}(\sqrt{2})$, $L = \mathbb{Q}(i)$ and M is defined in (b). Show that

$$[M : \mathbb{Q}] = [K : \mathbb{Q}][L : \mathbb{Q}].$$ ***(5 marks)***

2 (i) State the degrees formula for finite field extensions $K \subseteq L \subseteq M$. ***(2 marks)***

(ii) Let $L = \mathbb{Q}(\sqrt{2}, \sqrt[3]{2})$ and $\alpha = \sqrt{2} + \sqrt[3]{2}$.

(a) Find a $\mathbb{Q}$-basis of the field L. ***(12 marks)***

(b) Show that $L = \mathbb{Q}(\alpha)$. ***(5 marks)***

(c) Express the element $\dfrac{1}{\sqrt{2} + \sqrt[3]{2}}$ as a linear combination of the basis elements in the part (a). ***(6 marks)***

3 (i) Let b be an algebraic element over a field K.

(a) Give a definition of the minimal polynomial $m(x)$ of the element b over K. *(2 marks)*

(b) Prove that

$$[K(b) : K] = \deg m(x).$$

(10 marks)

(c) Find the minimal polynomial $m(x)$ of the element

$$\frac{1+\sqrt{2}}{1-\sqrt{2}}$$

over $\mathbb{Q}$. *(5 marks)*

(ii) State Gauss' Lemma. *(2 marks)*

(iii) Give a definition of the content $c(f)$ of a polynomial $f \in \mathbb{Z}[x]$ and using Gauss' Lemma show that

$$c(fg) = c(f)c(g)$$

for all $f, g \in \mathbb{Z}[x]$. *(6 marks)*

4 (i) Give a definition of a constructible point $P \in \mathbb{R}^2$. *(2 marks)*

(ii) State Standard Constructions I-IV (for each Standard Construction there is no need to give an algorithm for constructing points/lines). *(4 marks)*

(iii) Let $a, b \in \mathbb{R}$ be constructible real numbers. Show that the numbers

$$a-b, \quad a+b, \quad \frac{a}{b} \text{ (provided } b \neq 0) \text{ and } ab$$

are constructible. *(12 marks)*

(iv) Show that

$$\frac{3^{\frac{7}{8}}+\sqrt{2}}{\sqrt[4]{7}+17^{\frac{27}{64}}}$$

is constructible. *(4 marks)*

(v) Is $\sqrt[3]{7}$ a constructible number? Give a reason for your answer without going into detail. *(3 marks)*

End of Question Paper