The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2013–14**

Combinatorics **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) (a) State Pascal's Identity for binomial coefficients. ***(2 marks)***

(b) Using Pascal's Identity, or otherwise, show that

$$\sum_{m=r}^{s} \binom{m}{k} = \binom{s+1}{k+1} - \binom{r}{k+1}.$$

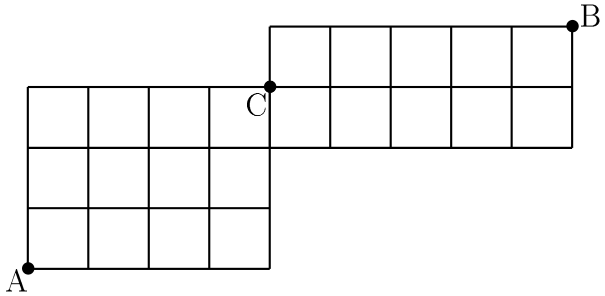
(5 marks)

(ii) Suppose there are $2n$ pupils, n from School A and n from School B. From these, we want to form a team consisting of n pupils, with one of the pupils from School A and one of the pupils from School B as the two leaders of the team. By considering the number of ways to do this, show that

$$\sum_{k=1}^{n-1} k(n-k)\binom{n}{k}^2 = n^2\binom{2n-2}{n-2}.$$

(9 marks)

(iii) This part of the question concerns routes in the grid illustrated:



(a) How many shortest routes are there from A to C along the lines of the grid? Give a brief reason for your answer. ***(3 marks)***

(b) Find the number of such routes from A to B. ***(6 marks)***

2 (i) Consider an $n \times n$ chess board with three of the corner squares removed. Show that the board cannot be completely covered by non-overlapping dominoes (that is, by pieces which cover exactly two adjacent squares). ***(6 marks)***

(ii) (a) State the Pigeon-hole Principle. ***(2 marks)***

(b) Show that if X is a set of eleven different integers from 1 to 150, there are two disjoint subsets of X with the same sum. ***(6 marks)***

(iii) (a) State the Inclusion/Exclusion Principle. ***(3 marks)***

(b) Use the Inclusion/Exclusion Principle to find the number of permutations of the numbers $1, 2, \ldots, 10$ such that no even number is fixed. ***(8 marks)***

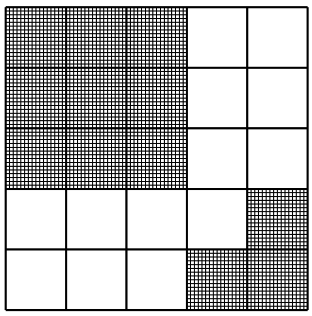
3 (i) Let B be part of an $n \times n$ board. Explain what is meant by the *rook polynomial* $r_B(x)$ of B. ***(1 mark)***

(ii) Let B be part of an $n \times n$ board and let s be one specified square of B. Let C be the board B with s deleted and let D be the board B with the whole of s's row and column deleted. Prove that

$$r_B(x) = r_C(x) + x r_D(x).$$

(8 marks)

(iii) Calculate the rook polynomial of (the unshaded part of) the board:



(7 marks)

(iv) State, with justification, whether the following are the scores of a tournament with 7 players.

$$6, 5, 4, 3, 2, 1, 0.$$

(3 marks)

(v) Let n be an even positive integer. Show that there can be a tournament of n players in which one player loses all their matches and all of the other players get the same score. State the scores. ***(6 marks)***

4 (i) State necessary and sufficient conditions for a $p \times q$ Latin rectangle to be extendable to an $n \times n$ Latin square. ***(2 marks)***

(ii) For what value of x can the following Latin rectangle be extended to a 6×6 Latin square?

$$\begin{pmatrix} 1 & 3 & 2 & 6 \\ 6 & 4 & 5 & x \\ 2 & 1 & 4 & 5 \\ 4 & 2 & 6 & 1 \end{pmatrix}$$

Write down one such extension. ***(8 marks)***

(iii) Let p be a prime number. Define $p \times p$ matrices A_k for $k = 1, 2, \ldots, p-1$ by: $(A_k)_{i,j}$ is the element of $\{1, 2, \ldots, p\}$ congruent to $ki + j$ mod p. You may assume that A_k is a Latin square, for $k = 1, 2, \ldots, p-1$. Show that A_k and A_h are orthogonal, for $1 \leq k, h \leq p-1$ and $k \neq h$. ***(6 marks)***

(iv) Consider a (v, b, r, k, λ) design. Give two equations expressing r in terms of the other parameters of the design. ***(3 marks)***

(v) Consider v varieties, where v is a positive integer, and make v blocks by assigning all the varieties except the ith one to block i. Show that the blocks make up a design and determine all the parameters of the design. ***(6 marks)***

End of Question Paper