data provided: list of formulae

MAS336

The University Of Sheffield.

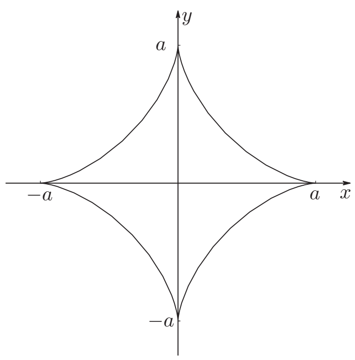
SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2013–14**

Differential Geometry **2 hours 30 minutes**

Answer ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

A list of formulae is provided on the last page.

1 (i) The curve in the following picture is parametrised by $(a\cos^3 t, a\sin^3 t)$ for some suitable values of t. Find its total arc length. ***(7 marks)***



(ii) Find the maximum and minimum values of curvature on the ellipse defined by $x^2 + 4y^2 = 1$. ***(9 marks)***

(iii) A curve on $\mathbb{R}^2$ is said to be *self-similar* if it is congruent to its own image under any map of the form

$$\varphi : \mathbb{R}^2 \to \mathbb{R}^2, \qquad \varphi(x, y) = (\lambda x, \lambda y), \quad \text{where } \lambda > 0.$$

Let C be a curve on $\mathbb{R}^2$ and $k(s)$ be its curvature function in terms of a unit-speed parameter $s > 0$. Prove that, if $k(\lambda s) = \lambda^{-1} k(s)$ for $s, \lambda > 0$, then C is self-similar. ***(9 marks)***

2 (i) Consider the following vector-valued function

$$\varphi(u,v) = \Big(u, \sqrt{v^2+1}, \ln(v+\sqrt{v^2+1})\Big).$$

(a) Show that φ defines a local isometry between $\mathbb{R}^2$ and some surface $S \subset \mathbb{R}^3$. ***(5 marks)***

(b) Find all the geodesics on S that pass through $(0,1,0)$. ***(7 marks)***

(c) What is the arc length of the shortest path on S between $(0,1,0)$ and $\big(1, \sqrt{2}, \ln(1+\sqrt{2})\big)$? ***(4 marks)***

(ii) Given a surface in $\mathbb{R}^3$, a curve on the surface is called a *normal section* if

- the curve is the intersection of the surface with a plane, and
- the normal vectors of the surface along the curve are parallel to the same plane.

Prove that any normal section is a pre-geodesic. ***(9 marks)***

3 (i) Let S be the surface (catenoid) parametrised by

$$\mathbf{x}(t,\theta) = (\cosh t \cos\theta, \cosh t \sin\theta, t).$$

(a) Show that the first fundamental form of this parametrisation is $\cosh^2 t\,(dt^2 + d\theta^2)$. ***(2 marks)***

(b) Find the area of the region on S between the latitudes $t=0$ and $t = \ln 2$. ***(7 marks)***

(c) Let S' be the cylinder in $\mathbb{R}^3$ defined by $x^2+y^2=1$. Find a function f such that the following map

$$\varphi : S \to S', \qquad \varphi(\mathbf{x}(t,\theta)) = (\cos\theta, \sin\theta, f(t))$$

preserves the areas of all regions. ***(8 marks)***

(ii) Let $X(u,v)$ and $Y(u,v)$ be two functions that satisfy

$$X_u(u,v) = Y_v(u,v), \qquad X_v(u,v) = -Y_u(u,v).$$

Show that the map $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\psi(u,v) = (X(u,v), Y(u,v))$ is conformal. ***(8 marks)***

4 (i) Suppose a surface in $\mathbb{R}^3$ has a parametrisation $\mathbf{x}(u,v)$ whose first and second fundamental forms are respectively

$$(u^2+1)^2(du^2+dv^2), \qquad 2du^2+(u^2-1)dv^2.$$

(a) Find the principal curvatures as functions of (u,v). ***(6 marks)***

(b) For what values of (u,v) are all tangential directions principal? ***(3 marks)***

(c) For what values of (u,v) are there tangential directions with zero normal curvature? ***(6 marks)***

(ii) Prove that, if a surface in $\mathbb{R}^3$ has a parametrisation whose second fundamental form is zero everywhere, then it must be contained in a plane. ***(10 marks)***

5 (i) The standard unit sphere S^2 can be parametrised by

$$\mathbf{x}(\phi,\theta) = (\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi)$$

whose first and second fundamental forms are both $d\phi^2 + \cos^2\phi\, d\theta^2$.

(a) Find the Gaussian curvature of S^2 as a function of (ϕ,θ). ***(5 marks)***

(b) Does there exist any local isometry between some part of S^2 and some part of a plane? Briefly explain why. ***(4 marks)***

(c) Find all the latitudes on S^2 (i.e. curves of the form $\phi =$ constant) along which parallel transport takes any tangent vector $\mathbf{v}$ to $-\mathbf{v}$. ***(7 marks)***

(ii) Let S be a surface whose Gaussian curvature is -1 everywhere. Prove that the area of any geodesic triangle on S is less than π. ***(9 marks)***

End of Question Paper

List of Formulae

For a curve on $\mathbb{R}^2$ parametrised by $\mathbf{x}(t) = (x(t), y(t))$:

- arc length from $\mathbf{x}(a)$ to $\mathbf{x}(b)$

$$\int_a^b ||\mathbf{x}'(t)||\, dt$$

- curvature

$$k(t) = \frac{x'(t)y''(t) - y'(t)x''(t)}{[x'(t)^2 + y'(t)^2]^{3/2}}$$

For a surface in $\mathbb{R}^3$ parametrised by $\mathbf{x}(u, v)$:

- first fundamental form

$$Edu^2 + 2Fdudv + Gdv^2, \quad E = \mathbf{x}_u \cdot \mathbf{x}_u, \quad F = \mathbf{x}_u \cdot \mathbf{x}_v, \quad G = \mathbf{x}_v \cdot \mathbf{x}_v$$

- surface areas

$$\iint \sqrt{EG - F^2}\, dudv$$

- second fundamental form

$$Ldu^2 + 2Mdudv + Ndv^2, \quad L = \mathbf{x}_{uu} \cdot \mathbf{n}, \quad M = \mathbf{x}_{uv} \cdot \mathbf{n}, \quad N = \mathbf{x}_{vv} \cdot \mathbf{n}$$

$$\text{where} \;\; \mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{||\mathbf{x}_u \times \mathbf{x}_v||}$$

- Weingarten matrix

$$W = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} L & M \\ M & N \end{bmatrix}$$

- Gaussian curvature

$$K = \det W$$

The Gauss-Bonnet formula for a compact region R on a surface:

$$\iint_R K dA + \int_{\partial R} k_g ds + \sum \text{turning angles} = 2\pi\chi(R)$$