The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn 2013-2014**

Financial Mathematics 2 **hours and** 30 **minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

Please leave this exam paper on your desk
Do not remove it from the hall

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to be completed by student

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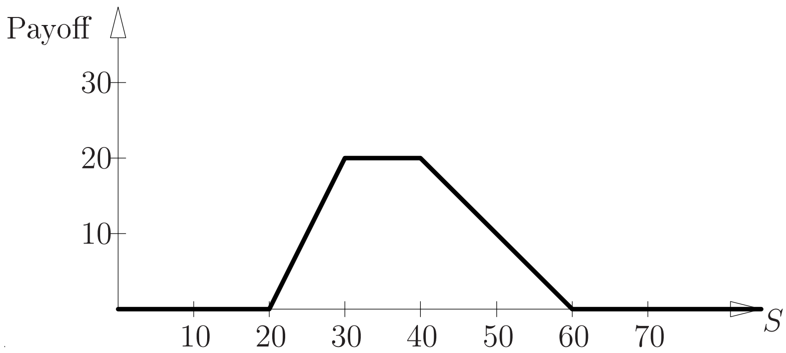
1 Consider a perpetual bond that pays £5 once a year, every year, and whose first payment occurs in 6 months. Assume that spot interest rates for all maturities are 3%.

(i) Find the price of the bond. ***(8 marks)***

(ii) Consider a N-year forward contract on the perpetual bond, where N is a positive integer. Show that the correct forward price in this forward contract is identical to the spot price of the bond. ***(8 marks)***

(iii) You are given the opportunity to take a long position in a two-year forward contract as in (ii) at a forward price of £160. Describe in detail an arbitrage opportunity available to you. ***(9 marks)***

2 (i) (a) Describe a portfolio consisting entirely of European put options on the same stock, with same expiration time $T > 0$, but with different strike prices, and whose payoff at time T as a function of S, the spot price of the stock at time T, is described by the graph below. ***(4 marks)***



(b) Let p_{20}, p_{30}, p_{40} and p_{60} be the prices of the above put options with strike prices $20, 30, 40$ and 60, respectively, and let c_{10} be the price of a European call option on the same stock, with expiration at time T and with strike price 10. By comparing the payoff of the portfolio in (a) and the payoff of the call option above, describe an inequality involving $c_{10}, p_{20}, p_{30}, p_{40}$ and p_{60}. ***(5 marks)***

(ii) Describe what American put options are. ***(3 marks)***

(iii) The price of a stock which pays no dividends is currently £20. Over each of the next three 1-year periods the stock price will either increase by 10% or decrease by 10%. Suppose that all interest rates are constant and equal to 3%.

(a) Use a binomial tree to find the price of a three-year American put option on this stock with strike price £20. ***(11 marks)***

(b) Describe all circumstances when a rational investor should exercise the option described in (a) before its expiration. ***(2 marks)***

3 (i) Consider a derivative on a stock which entitles the holder to one payoff at time T; the amount of this payoff is £1 if the stock price S_T at time T is at least a, for some positive number a, and zero otherwise. Let S be the price of the stock and assume, as usual, that S follows the process

$$dS = \mu S dt + \sigma S dB$$

for constants μ and $\sigma > 0$ and where B is a Brownian motion. Assume further that all interest rates are constant and equal to r.

(a) Use Ito's Lemma to show that $\log S$ follows the process

$$d(\log S) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dB.$$ ***(6 marks)***

(b) Find an expression for the probability *in a risk-neutral world* of the event $S_T \leq a$. ***(8 marks)***

(c) Apply a risk-neutral valuation argument to show that, for any $0 \leq t \leq T$, the value of this derivative equals

$$e^{-r(T-t)}\Phi\left(\frac{\log\left(S_t/a\right) + \left(r - \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}\right),$$

where Φ is the cumulative distribution function of the standard normal distribution. ***(3 marks)***

(ii) (a) Verify that $f(S,t) = e^{(2r+3\sigma^2)(T-t)}S^3$ is a solution of the Black-Scholes partial differential equation

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf.$$

(4 marks)

(b) Consider a derivative with underlying asset whose price S follows the Ito process $dS = \mu S dt + \sigma S dB$ and which provides a single payoff at time $T > 0$ in the amount of S_T^3, where S_T is the underlying asset price at time T. What is the price of this derivative at time $0 \leq t < T$? ***(4 marks)***

4 (i) Define the following concepts in the context of Portfolio Theory.

(a) The *market portfolio.* ***(2 marks)***

(b) The *capital market line.* ***(2 marks)***

(c) The *beta-coefficient* of an investment. ***(2 marks)***

(ii) Consider a market with risk-free return r_B and whose market portfolio M has expected return r_M and standard deviation of returns σ_M. Let A be an investment with expected return of r_A, standard deviation of returns σ_A and beta coefficient β.

(a) What is the slope of the capital market line? ***(2 marks)***

(b) Show that the market portfolio is the unique portfolio P which maximizes

$$\frac{r_P - r_B}{\sigma_P}$$

as P ranges over all portfolios consisting entirely of risky investments. ***(3 marks)***

(c) Describe parametrically the curve c in the σ-r plane consisting of all points corresponding to investments spread between A and M. ***(6 marks)***

(d) Explain why c is tangent to the capital market line at the point M. ***(3 marks)***

(e) Use (d) to show that

$$r_A = \beta(r_M - r_B) + r_B.$$ ***(5 marks)***

End of Question Paper