Data provided: Formula sheet

The University Of Sheffield.

MAS381

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2013–14**

Mathematics III (Electrical) **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) Find the singularities and their orders for the functions

$$(a)\ \frac{1}{z^2+1}, \quad (b)\ \frac{1}{z^4+4z^2+4}, \quad (c)\ \frac{(z+1)^3}{z^2+2z+2}$$

(8 marks)

(ii) Determine the constant a such that the function $u(x, y) = e^{3x} \cos ay$ is harmonic and hence, find a harmonic conjugate, $v(x, y)$. With the help of the two functions find an analytic complex function of the form $f(z) = u+jv$ that satisfies $f(0) = 1$. ***(17 marks)***

2 (i) Find the Laurent series of the function

$$f(z) = \frac{1}{z(z-1)(z-2)}$$

valid for $1 < |z| < 2$. ***(12 marks)***

(ii) For the function $w = 1/z$, find the image in the w plane of the following:

(a) The point $z = 1 - j$, ***(2 marks)***

(b) The line $z = -1 + re^{j\pi/2}, \quad 0 \le r < \infty.$ ***(11 marks)***

Sketch your results in the z and w planes.

3 (i) Find all the poles of the function $f(z) = \dfrac{e^{3z}}{z^2(z^2+2z+2)}$ and plot them on an Argand diagram. Hence evaluate the integral $\oint_C f(z)\,dz$, writing your solutions in the form $a + jb$ where a and b are real, where

(a) C is the circle $|z| = 2$

(b) C is the circle $|z-2| = 1$. ***(13 marks)***

(ii) By constructing a suitable contour in the complex plane, use the method of residues to evaluate the real integral

$$I = \int_{-\infty}^{\infty} \frac{1}{1+x^2}dx$$

(12 marks)

4 (i) Let S be the surface consisting of the hemisphere S_1 given, in spherical coordinates, by $r = 1$, $0 \le \phi \le 2\pi$, $0 \le \theta \le \pi/2$ together with the disc S_2 given by $0 \le r \le 1$, $0 \le \phi \le 2\pi$, $\theta = \pi/2$. Let V be the hemispherical volume enclosed by S and let $\mathbf{E}$ be the vector field

$$\mathbf{E} = x\mathbf{i} + y\mathbf{j} + (1-z)\mathbf{k}$$

(a) Evaluate the integral

$$\iint_S \mathbf{E} \cdot d\mathbf{S}$$

(18 marks)

(b) Evaluate

$$\iiint_E div(\mathbf{E})dV$$

and verify that

$$\iiint_V div(\mathbf{E})dV = \iint_S \mathbf{E} \cdot d\mathbf{S}$$

(7 marks)

End of Question Paper

Formula sheet

- The general formula for the residue at a pole z_0, of order m is

$$\frac{1}{(m-1)!}\lim_{z\to z_0}\left\{\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]\right\}$$

- Useful identities

$$\sin 2\theta = 2\sin\theta\cos\theta, \quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

- The spherical volume element is given by

$$dV = r^2 \sin\phi \; dr \; d\phi \; d\theta$$

- The position vector in spherical coordinates for a unit radius is given as

$$\mathbf{r} = (x, y, z) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$