The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2013–2014

Topics in Advanced Fluid Mechanics — **2 hours 30 minutes**

Marks will be awarded for your best ***four*** *answers.*

1 We consider the 3D Navier-Stokes equations for an incompressible fluid of a constant unit density:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p + \nu\triangle\boldsymbol{u},$$
$$\nabla\cdot\boldsymbol{u} = 0.$$

(1) Recast the above equations in the following form

$$\frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{u}\times(\nabla\times\boldsymbol{u}) - \nabla\left(p + \frac{|\boldsymbol{u}|^2}{2}\right) + \nu\triangle\boldsymbol{u}.$$

Hint: $\nabla(\boldsymbol{A}\cdot\boldsymbol{B}) = \boldsymbol{A}\times(\nabla\times\boldsymbol{B}) + \boldsymbol{B}\times(\nabla\times\boldsymbol{A}) + (\boldsymbol{B}\cdot\nabla)\boldsymbol{A} + (\boldsymbol{A}\cdot\nabla)\boldsymbol{B}$ ***(4 marks)***

(2) Derive a set of equations for the impulse variable $\boldsymbol{\gamma} = \boldsymbol{u} + \nabla\phi$ as

$$\frac{\partial \boldsymbol{\gamma}}{\partial t} = \boldsymbol{u}\times(\nabla\times\boldsymbol{\gamma}) + \nabla\Lambda + \nu\triangle\boldsymbol{\gamma}, \tag{1.1}$$

$$\frac{\partial \phi}{\partial t} = p + \frac{|\boldsymbol{u}|^2}{2} + \Lambda + \nu\triangle\phi,$$

by suitably defining a scalar function $\Lambda(\boldsymbol{x}, t)$. ***(6 marks)***

(3) Consider a flow for which

$$\boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{u}) = 0$$

holds everywhere and at any time. By separating (1.1) into solenoidal and potential parts, derive the equations for $\boldsymbol{u}$ and ϕ.

Hint: If a vector is simultaneously solenoidal (i.e. divergence-free) and potential (i.e. curl-free), then it is a zero vector. ***(11 marks)***

(4) Prove that $p + \frac{|\boldsymbol{u}|^2}{2}$ is independent of spatial coordinates $\boldsymbol{x}$ and also $\boldsymbol{u}$ solves heat diffusion equations. ***(4 marks)***

2 We consider the Burgers equation for an inviscid fluid in $\mathbb{R}^1$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \tag{2.1}$$

for an initial condition $u(x,0) = u_0(x)$.

(1) Determine a path of a fluid particle which is at $x = a$ initially. Hence obtain an implicit solution

$$u(x,t) = u_0(x - tu(x,t)), \qquad (2.2).$$

(6 marks)

(2) By differentiating (2.2) with respect to x, obtain a solution for $\dfrac{\partial u}{\partial x}$. ***(7 marks)***

(3) For $u_0(x) = -\sin x$, consider Fourier series

$$a = x + \sum_{n=1}^{\infty} A_n \sin(nx).$$

and carry out the integrations to determine A_n explicitly ($n \geq 1$). ***(6 marks)***

(4) Derive a Fourier representation for $u(x,t)$ using the Bessel function.

(6 marks)

Hints:
The Fourier coefficients are given by

$$A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx$$

for a Fourier sine series $f(x) = \sum_{n=1}^{\infty} A_n \sin(nx)$.

The Bessel function of the first kind $J_n(z)$ is defined by

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - z\sin\theta)d\theta.$$

3 Consider a model equation for vorticity ω defined in $\mathbb{R}^1$:

$$\frac{\partial \Omega}{\partial t} = \Omega H[\Omega]$$

with an initial condition

$$\Omega(x, t = 0) = \Omega_0(x).$$

Here $H[\Omega]$ denotes the Hilbert transform on $\mathbb{R}^1$

$$H[\Omega](x) = \frac{1}{\pi} ⨍_{-\infty}^{\infty} \frac{\Omega(y)}{x - y} dy,$$

where $⨍$ is a principal-value integral.

(1) Show that $\Omega = A$, where A is a constant independent of x, is a steady solution. ***(4 marks)***

(2) By decomposing as $\Omega(x, t) = A + \omega(x, t)$, derive an equation for the fluctuation part ω. ***(4 marks)***

(3) Assuming that ω is very small ($|\omega| \ll |A|$), obtain a linearised equation for ω

$$\frac{\partial \omega}{\partial t} = AH[\omega]. \tag{3.1}$$

(2 marks)

(4) Solve the equation (3.1) explicitly for $\omega(x, t = 0) = \sin x$. ***(15 marks)***

4 We consider a system of N point-vortices subject to an external field

$$U = ax - by, \; V = bx - ay,$$

where a, b are constants. The equations for the point vortices in this case read

$$\frac{dx_i}{dt} = \frac{-1}{2\pi} \sum_{j=1}^{N}{}' \kappa_j \frac{y_i - y_j}{(x_i - x_j)^2 + (y_i - y_j)^2} + U,$$

$$\frac{dy_i}{dt} = \frac{1}{2\pi} \sum_{j=1}^{N}{}' \kappa_j \frac{x_i - x_j}{(x_i - x_j)^2 + (y_i - y_j)^2} + V,$$

where $\sum'$ denotes a summation excluding $j = i$. Here (x_i, y_i), $i = 1, 2, \ldots, N$ denotes coordinates of a point vortex of strength κ_i.

(1) Assume that there is only one point vortex ($N = 1$), whose position is (x, y). Determine the locus of the point vortex, that is, derive an equation for $\dfrac{dy}{dx}$ and solve it. (It is *not* necessary to obtain x, y as functions of time.) ***(8 marks)***

(2) Consider the case of two point vortices ($N = 2$) with $\kappa_1 = \kappa_2 = 2\pi$ for simplicity. Let (x_1, y_1) and (x_2, y_2) be their positions and write down the equations for x_1, x_2, y_1 and y_2. ***(3 marks)***

(3) Consider a set of new variables $x = x_1 - x_2$ and $y = y_1 - y_2$ and derive equations for $\dfrac{dy}{dx}$. ***(4 marks)***

(4) Determine the locus of x, y by solving the equation derived in (3). ***(8 marks)***

(5) What is the shape of the locus when $a = 0$? ***(2 marks)***

5 The motion of a vortex layer of uniform strength is governed by

$$\frac{\partial z(\alpha,t)^*}{\partial t} = -\frac{\mathrm{i}}{2\pi} \fint_{-\infty}^{\infty} \frac{1}{z(\alpha,t)-z(\beta,t)} d\beta, \tag{5.1}$$

where

$$z(\alpha,t) = x(\alpha,t) + iy(\alpha,t)$$

denotes the position of a fluid particle α on the layer, $*$ complex conjugate and $\fint$ a principal-value integral. We study its stability property of a flat state $z_0(\alpha) = \alpha$.

(1) By setting $z(\alpha) = \alpha + if(\alpha,t)$, where $f(\alpha,0) = 0$, derive from (5.1)

$$\frac{\partial f(\alpha)^*}{\partial t} = \frac{1}{2\pi} \fint_{-\infty}^{\infty} \frac{d\beta}{\alpha - \beta + i(f(\alpha) - f(\beta))}, \tag{5.2}$$

(3 marks)

(2) Derive a linearised equation from (5.2)

$$\frac{\partial f(\alpha)^*}{\partial t} = -\frac{i}{2\pi} \fint_{-\infty}^{\infty} \frac{f(\alpha)-f(\beta)}{(\alpha-\beta)^2} d\beta. \tag{5.3}$$

(6 marks)

(3) Rewrite (5.3) as

$$\frac{\partial f^*}{\partial t} = -\frac{i}{2} H\left[\frac{\partial f}{\partial \alpha}\right], \tag{5.4}$$

using integration by parts. See Question 3 for the definition of the Hilbert transform H. ***(5 marks)***

(4) Derive from (5.4)

$$\frac{\partial^2 f}{\partial t^2} = -\frac{1}{4}\frac{\partial^2 f}{\partial \alpha^2}.$$

(6 marks)

(5) Consider a Fourier mode of the form

$$f \propto \exp(ik\alpha + \lambda t)$$

to determine its linear stability, where k is the wavenumber and λ the growth rate. ***(5 marks)***

End of Question Paper