The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS Autumn Semester 2013-2014

Analytical Dynamics and Classical Field Theory **3 Hours**

Answer five questions. If you answer more than five questions, only your best five will be counted.

1 In this question, we consider a particle of mass m moving in one dimension (denoted by q) under the influence of a potential $V(q)$.

(i) Write down the Euler–Lagrange equation and define every quantity in this equation. ***(5 marks)***

(ii) Suppose that the Lagrange–function L does not depend explicitly on time t. Show that in this case

$$L - \dot{q}\frac{\partial L}{\partial \dot{q}} = \text{constant}$$

(9 marks)

(iii) Show that the total energy of the particle is conserved. ***(6 marks)***

2 A particle of mass m is attached to a string of fixed length k. The particle swings under the influence of gravity, forming a spherical pendulum. The Lagrange function of this system is given by

$$L(\theta, \phi, \dot{\theta}, \dot{\phi}) = \frac{1}{2} mk^2 \left(\dot{\theta}^2 + \sin^2\theta \dot{\phi}^2 \right) + mgk\cos\theta \ .$$

(i) Write down the two Euler–Lagrange equations for the system and use them to show that

$$mk^2 \dot{\phi} \sin^2\theta = \text{constant} = C$$

and that

$$\ddot{\theta} + \frac{g}{k} \sin\theta = \left(\frac{C}{mk^2} \right)^2 \cot\theta \ \text{cosec}^2\theta$$

(11 marks)

(ii) Show that the solution $\theta = \theta_0 = \text{constant}$ is only possible if

$$\frac{g}{k} \left(\frac{mk^2}{C} \right)^2 \sin^4\theta_0 = \cos\theta_0 \ .$$

Explain why $\theta_0 \leq \pi/2$. ***(6 marks)***

(iii) Without writing down the Hamilton function, explain why it is constant and represents the total energy of the system. ***(3 marks)***

3 Consider a single particle of mass m, whose dynamics is described by generalised coordinates $q_i = x, y, z$ and generalised momenta $p_i = p_x, p_y, p_z$.

(i) Define the Poisson brackets [f,g] between two functions $f(q_i, p_i)$ and $g(q_i, p_i)$. Prove that for any function f depending on q_i and p_i, the following equations hold:

$$[f(q_i, p_i), p_x] = \frac{\partial f}{\partial x}, \quad [f(q_i, p_i), p_y] = \frac{\partial f}{\partial y}, \quad [f(q_i, p_i), p_z] = \frac{\partial f}{\partial z}$$

(5 marks)

(ii) The angular momentum of the particle is defined as $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, where $\mathbf{r} = (x, y, z)$ and $\mathbf{p} = (p_x, p_y, p_z)$. Show that

$$[L_z, x] = y, \quad [L_z, y] = -x, \quad [L_z, z] = 0$$

and

$$[L_z, p_x] = p_y, \quad [L_z, p_y] = -p_x, \quad [L_z, p_z] = 0$$

(15 marks)

4 (i) State Hamilton's principle and define all quantities appearing in this principle. ***(3 marks)***

(ii) Let $F[y, y', x]$ be a function dependent on $y(x), y'(x) = dy/dx$ and x. Show that making the functional $G = \int_A^B F[y, y', x]dx$ stationary with the boundary conditions fixed at A and B, leads to the Euler–Lagrange equation for $F[y, y', x]$

$$\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{dy} = 0 \ . \tag{1}$$

(9 marks)

(iii) Using (1), show that the shortest path between two points in the xy-plane is a straight line. ***(8 marks)***

5 (i) You are given the four–vector z^μ. What is the definition of z_μ? How do z^μ and z_μ transform under Lorentz–transformations? Let $\sigma^{\mu\nu}$ transform like a tensor under Lorentz–transformations and b_μ a contravariant four-vector. Show that $\sigma^{\mu\nu} b_\mu$ transforms like a four–vector. ***(6 marks)***

(ii) A particle moves in Minkowski spacetime. Define the proper–time, the four-velocity and the four-acceleration. Furthermore, show that $a^\mu u_\mu = 0$. ***(8 marks)***

(iii) A particle is moving under the influence of an electromagnetic field, described by the field strength tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. The equation of motion is

$$m\frac{du^\mu}{d\tau} = \frac{q}{c}F^{\mu\nu}u_\nu,$$

where m is the mass of the particle and q its charge. Use the equation of motion to show that if $u^\mu u_\mu = -c^2$ at $\tau = 0$, then $u^\mu u_\mu = -c^2$ for all τ. ***(6 marks)***

6 (i) In words, describe the contents of Noether's theorem. Give two examples of a conserved quantity and the corresponding symmetry. ***(5 marks)***

(ii) The action describing the electromagnetic field in the presence of an electromagnetic current is described by the action

$$S = \int d^4x \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + j^\mu A_\mu \right) \ ,$$

where A^μ is the electromagnetic four–vector, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and j^μ the four–current. Show that under the gauge–transformation $A_\mu \to A_\mu + \partial_\mu \theta$, where θ is a generic function of the space-time coordinates, the action above is invariant if and only if $\partial_\mu j^\mu = 0$. Assume that you can disregard any boundary terms at infinity (i.e. that θ vanishes at infinity). Hint: you don't need the formula of the Noether current for this question. ***(10 marks)***

(iii) Suppose that the photon has a mass m. In this case, in the action above we would have to add a term

$$\int d^4x \left(\frac{1}{2} m^2 A^\nu A_\nu \right) \ .$$

Verify that for $j^\mu = 0$ the action is not invariant under the gauge transformation.

(5 marks)

End of Question Paper