The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2013–14**

Analytic Number Theory **2 hours 30 minutes**

*Answer **Question 1** and three other questions. You are advised **not** to answer more than three of the questions 2 to 5: if you do, only your best three will be counted.*

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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1 (i) Given a character χ of the group $(\mathbb{Z}/N\mathbb{Z})^\times$, explain how one defines the associated *Dirichlet L-function* $L(s,\chi)$ and indicate its region of convergence. Describe, without proof, the behaviour of $L(1,\chi)$ and $\lim_{\sigma\to 1+} \sum \frac{\chi(p)}{p^\sigma}$.

(7 marks)

(ii) This question asks you to illustrate the proof of Dirichlet's Theorem in a specific case.

(a) List the characters of $(\mathbb{Z}/8\mathbb{Z})^\times$, indicating which are the non-trivial characters. ***(5 marks)***

(b) Prove that $0 < L(1,\chi) < 2$ for each non-trivial character χ on your list. ***(6 marks)***

(c) Prove that there are infinitely many primes congruent to $3 \pmod 8$.

(7 marks)

2 (i) State the Prime Number Theorem. ***(1 mark)***

Use the Prime Number Theorem to show that $\log \pi(x) \sim \log x$. Deduce that $x \sim \pi(x)\log\pi(x)$, and hence show that p_n, the n-th prime, is asymptotic to $n\log n$. ***(10 marks)***

(ii) Let $\Phi(X)$ be the polynomial $X^{18}+X^9+1$. Note that $\Phi(X) = \frac{X^{27}-1}{X^9-1}$.

(a) Show that the set of primes p for which there is an integer n with p a factor of $\Phi(n)$ is infinite. ***(4 marks)***

(b) Now let a be an integer, and let $p > 3$ be a prime dividing $\Phi(a)$. Show that p divides $a^{27}-1$, and that the order of a in $(\mathbb{Z}/p\mathbb{Z})^\times$ is 27. ***(6 marks)***

(c) Deduce that there are infinitely many primes of the form $27k+1$.

(4 marks)

3 (i) Write down a formula for the highest power of a prime that divides $n!$. Find all positive integers n for which the decimal expansion of $n!$ ends in exactly 501 zeroes. ***(8 marks)***

(ii) Let $g(t) := \frac{\log 2}{6} \frac{t^2}{\log t} - t - 1$.

(a) Show that $\frac{t}{\log t}$ is increasing when $t \geq 3$. ***(2 marks)***

(b) Calculate $g'(t)$ and show that it is increasing when $t \geq 3$. Verify that

$$g'(2^4) = \frac{2\log 2 - 1}{6\log 2},$$

and hence conclude that $g(t)$ is increasing when $t \geq 16$. Also, verify that $g(2^5) = \frac{17}{15}$. ***(10 marks)***

(c) Now assume that the inequality

$$\pi(2n) - \pi(n) \geq \frac{\log 2}{3} \frac{2n}{\log 2n} - \sqrt{2n} - 1$$

holds for all $n \geq 3$. Show that if $n \geq 2^9$ then there are primes p and q such that $n < p < q < 2n$. ***(5 marks)***

4 (i) Let $f, g, h : \mathbb{N} \longrightarrow \mathbb{C}$ be three arithmetic functions related by the following formal product of Dirichlet series:

$$\left(\sum_{n=1}^{\infty} \frac{f(n)}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{g(n)}{n^s}\right) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}.$$

Write down the relation between f, g and h. Write down the (formal) Euler product expansion of $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ if f is a multiplicative arithmetic function, and give a simplified form when f is totally multiplicative. ***(5 marks)***

(ii) Show formally that

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}.$$

Here, $\phi(n)$ is the number of positive integers less than or equal to n and coprime to n. You may assume that ϕ is multiplicative. ***(6 marks)***

(iii) Define the arithmetic function $P : \mathbb{N} \longrightarrow \mathbb{Z}$ by setting $P(n)$ to be the number of distinct primes dividing n. Thus $P(1) = 0, P(4) = 1, P(12) = 2$ etc.

(a) Show that $P(mn) = P(m) + P(n)$ if m, n are co-prime positive integers. ***(2 marks)***

(b) Let N be a positive integer. Show that the number of pairs (m, n) of co-prime positive integers m, n satisfying $mn = N$ is $2^{P(N)}$. ***(4 marks)***

(c) Show that

$$\sum_{n=1}^{\infty} \frac{2^{P(n)}}{n^s} = \frac{\zeta(s)^2}{\zeta(2s)}.$$

Assuming that the above relation holds for $\mathrm{Re}(s) > 1$, evaluate the series

$$\sum_{\substack{m,n=1 \\ (m,n)=1}}^{\infty} \frac{1}{m^2 n^2}$$

numerically. You may assume that $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(4) = \frac{\pi^4}{90}$. ***(8 marks)***

5 (i) Let $f : \mathbb{N} \longrightarrow \mathbb{C}$ be an arithmetic function, and let

$$F(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

What can you say about the analyticity of $F(s)$ if the sequence

$$f(1), f(1) + f(2), f(1) + f(2) + f(3), \ldots$$

is bounded? Deduce that if $\sum_{n=1}^{\infty} \frac{f(n)}{n^{s_0}}$ converges at $s_0 \in \mathbb{C}$, then $F(s)$ is analytic in the half-plane $\mathrm{Re}(s) > \mathrm{Re}(s_0)$. ***(5 marks)***

Recall that the *Riemann zeta function* $\zeta(s)$ is defined for $\mathrm{Re}(s) > 1$ by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

(a) Write down the *Euler product* for $\zeta(s)$, indicating clearly in what region of the complex plane the formula is valid. ***(2 marks)***

(b) Derive a relation between $\zeta(s)$ and the series

$$1 + \frac{1}{2^s} - \frac{2}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} - \frac{2}{6^s} + \cdots ,$$

and explain how you could use this to extend the definition of $\zeta(s)$ to $\mathrm{Re}(s) > 0$. You should indicate the possible poles of $\zeta(s)$ under this extension. ***(10 marks)***

(ii) Recall that $B_n(x)$, the n-th Bernoulli polynomial, is defined by the generating series

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

Show that $B_n(x+1) - B_n(x) = nx^{n-1}$ for all $n \geq 0$. Deduce that

$$1^{n-1} + 2^{n-1} + \ldots + N^{n-1} = \frac{B_n(N+1) - B_n(1)}{n}$$

for all integers $n, N \geq 1$. ***(8 marks)***

End of Question Paper