The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2013-14**

Fields **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) For each of the subsets J_1, J_2 of $\mathbb{C}$ specified below determine, with justification, whether it is a subfield of $\mathbb{C}$:

(a) $J_1 = \{a + bi\sqrt{7} : \ a, b \in \mathbb{Q}\}$, ***(5 marks)***

(b) $J_2 = \{a + b\sqrt{3} + ci + di\sqrt{-3} : \ a, b, c, d \in \mathbb{Q}\}$. ***(4 marks)***

(ii) (a) Explain what is meant by saying that a field K is a *finite field extension* of $\mathbb{Q}$. ***(2 marks)***

(b) Let K and L be finite field extensions of $\mathbb{Q}$ such that $K, L \subseteq \mathbb{C}$, and M be the subfield of $\mathbb{C}$ generated by K and L. Show that M is a finite field extension of $\mathbb{Q}$ and that

$$[M : \mathbb{Q}] \leq [K : \mathbb{Q}][L : \mathbb{Q}].$$ ***(9 marks)***

(c) Suppose that $K = \mathbb{Q}(\sqrt{2})$, $L = \mathbb{Q}(i)$ and M is defined in (b). Show that

$$[M : \mathbb{Q}] = [K : \mathbb{Q}][L : \mathbb{Q}].$$ ***(5 marks)***

2 (i) State the degrees formula for finite field extensions $K \subseteq L \subseteq M$. ***(2 marks)***

(ii) Let $L = \mathbb{Q}(\sqrt{2}, \sqrt[3]{2})$ and $a = \sqrt{2} + \sqrt[3]{2}$.

(a) Find a $\mathbb{Q}$-basis of the field L. ***(12 marks)***

(b) Show that $L = \mathbb{Q}(a)$. ***(5 marks)***

(c) Express the element $\dfrac{1}{\sqrt{2} + \sqrt[3]{2}}$ as a linear combination of the basis elements in the part (a). ***(6 marks)***

3 (i) Let b be an algebraic element over a field K.

(a) Give a definition of the minimal polynomial $m(x)$ of the element b over K. *(2 marks)*

(b) Prove that
$$[K(b) : K] = \deg m(x).$$ *(10 marks)*

(c) Find the minimal polynomial $m(x)$ of the element
$$\frac{1+\sqrt{2}}{1-\sqrt{2}}$$
over $\mathbb{Q}$. *(5 marks)*

(ii) Let L be a field and $\sigma : L \to L$ be an automorphism of L.

(a) Show that $L^\sigma := \{a \in L \mid \sigma(a) = a\}$ is a subfield of L (the *fixed* field of σ). *(6 marks)*

(b) Let σ be the complex conjugation of the field of complex numbers $\mathbb{C}$. Find $\mathbb{C}^\sigma$. *(2 marks)*

4 Let $K \subseteq L$ be fields.

(i) Define the group $G(L/K)$. *(2 marks)*

(ii) Find the group $G(\mathbb{Q}(\sqrt{5})/\mathbb{Q})$. *(8 marks)*

(iii) Find the group $G(\mathbb{F}_5(\sqrt{\overline{2}})/\mathbb{F}_5)$ where $\mathbb{F}_5 = \{\overline{0}, \overline{1}, \ldots, \overline{4}\}$ is the field that contains 5 elements and $\overline{2} \in \mathbb{F}_5$. *(10 marks)*

(iv) Let F be a finite field. Prove that every homomorphism $\sigma : F \to F$ is an automorphism. *(5 marks)*

End of Question Paper