The University Of Sheffield.

**SCHOOL OF MATHEMATICS AND STATISTICS** **Autumn 2013-2014**

**Financial Mathematics** $2$ **hours and** $30$ **minutes**

*Attempt all the questions. The allocation of marks is shown in brackets.*

**Please leave this exam paper on your desk**
**Do not remove it from the hall**

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to be completed by student

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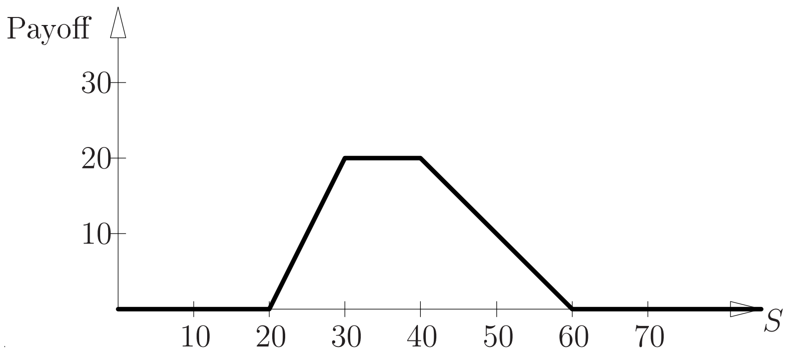
**1** Consider a perpetual bond that pays £5 once a year, every year, and whose first payment occurs in 6 months. Assume that spot interest rates for all maturities are 3%.

(i) Find the price of the bond. ***(8 marks)***

(ii) Consider a $N$-year forward contract on the perpetual bond, where $N$ is a positive integer. Show that the correct forward price in this forward contract is identical to the spot price of the bond. ***(8 marks)***

(iii) You are given the opportunity to take a long position in a two-year forward contract as in (ii) at a forward price of £160. Describe in detail an arbitrage opportunity available to you. ***(9 marks)***

**2** (i) (a) Describe a portfolio consisting entirely of European put options on the same stock, with same expiration time $T > 0$, but with different strike prices, and whose payoff at time $T$ as a function of $S$, the spot price of the stock at time $T$, is described by the graph below. ***(4 marks)***




(b) Let $p_{20}, p_{30}, p_{40}$ and $p_{60}$ be the prices of the above put options with strike prices $20, 30, 40$ and 60, respectively, and let $c_{10}$ be the price of a European call option on the same stock, with expiration at time $T$ and with strike price 10. By comparing the payoff of the portfolio in (a) and the payoff of the call option above, describe an inequality involving $c_{10}, p_{20}, p_{30}, p_{40}$ and $p_{60}$. ***(5 marks)***

(ii) Describe what American put options are. ***(3 marks)***

(iii) The price of a stock which pays no dividends is currently £20. Over each of the next three 1-year periods the stock price will either increase by 10% or decrease by 10%. Suppose that all interest rates are constant and equal to 3%.

(a) Use a binomial tree to find the price of a three-year American put option on this stock with strike price £20. ***(11 marks)***

(b) Describe all circumstances when a rational investor should exercise the option described in (a) before its expiration. ***(2 marks)***

**3** (i) Consider a derivative on a stock which entitles the holder to one payoff at time $T$; the amount of this payoff is £1 if the stock price $S_T$ at time $T$ is at least $a$, for some positive number $a$, and zero otherwise. Let $S$ be the price of the stock and assume, as usual, that $S$ follows the process

$$dS = \mu S dt + \sigma S dB$$

for constants $\mu$ and $\sigma > 0$ and where $B$ is a Brownian motion. Assume further that all interest rates are constant and equal to $r$.

(a) Use Ito's Lemma to show that $\log S$ follows the process

$$d(\log S) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dB.$$ ***(6 marks)***

(b) Find an expression for the probability *in a risk-neutral world* of the event $S_T \leq a$. ***(8 marks)***

(c) Apply a risk-neutral valuation argument to show that, for any $0 \leq t \leq T$, the value of this derivative equals

$$e^{-r(T-t)}\Phi\left(\frac{\log\left(S_t/a\right) + \left(r - \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}\right),$$

where $\Phi$ is the cumulative distribution function of the standard normal distribution. ***(3 marks)***

(ii) (a) Verify that $f(S,t) = e^{(2r+3\sigma^2)(T-t)}S^3$ is a solution of the Black-Scholes partial differential equation

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf.$$

***(4 marks)***

(b) Consider a derivative with underlying asset whose price $S$ follows the Ito process $dS = \mu S dt + \sigma S dB$ and which provides a single payoff at time $T > 0$ in the amount of $S_T^3$, where $S_T$ is the underlying asset price at time $T$. What is the price of this derivative at time $0 \leq t < T$? ***(4 marks)***

**4** (i) Define the following concepts in the context of Portfolio Theory.

(a) The *market portfolio*. ***(2 marks)***

(b) The *capital market line*. ***(2 marks)***

(c) The *beta-coefficient* of an investment. ***(2 marks)***

(ii) Consider a market with risk-free return $r_B$ and whose market portfolio $M$ has expected return $r_M$ and standard deviation of returns $\sigma_M$. Let $A$ be an investment with expected return of $r_A$, standard deviation of returns $\sigma_A$ and beta coefficient $\beta$.

(a) What is the slope of the capital market line? ***(2 marks)***

(b) Show that the market portfolio is the unique portfolio $P$ which maximizes

$$\frac{r_P - r_B}{\sigma_P}$$

as $P$ ranges over all portfolios consisting entirely of risky investments. ***(3 marks)***

(c) Describe parametrically the curve $c$ in the $\sigma$-$r$ plane consisting of all points corresponding to investments spread between $A$ and $M$. ***(6 marks)***

(d) Explain why $c$ is tangent to the capital market line at the point $M$. ***(3 marks)***

(e) Use (d) to show that

$$r_A = \beta(r_M - r_B) + r_B.$$ ***(5 marks)***

## End of Question Paper