The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS Spring Semester 2013-2014

FOUNDATION YEAR MATHEMATICS II 3 hours

Attempt all the questions. The allocation of marks is shown in brackets.

1. (i) (a) Let θ be an angle such that $90^{\circ} \leq \theta \leq 180^{\circ}$ and that $\sin\theta = \frac{3}{5}$, find the values of $\cos\theta$, $\tan\theta$ and $\sin 2\theta$.

(4 marks)

(b) Using the identities $\sin(A-B) = \sin A\cos B - \cos A\sin B$ and $\cos(A-B) = \cos A\cos B + \sin A\sin B$, prove that

$$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B}.$$

(4 marks)

(ii) Let $\vec{OA} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $\vec{OB} = 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$.

(a) Determine, in degrees to an accuracy of one decimal place, the acute angle between $\vec{OA}$ and $\vec{OB}$.

(4 marks)

(b) Calculate the area of the triangle OAB to an accuracy of 1 decimal place.

(4 marks)

(c) Find a non-zero vector that is perpendicular to both $\vec{OA}$ and $\vec{OB}$.

(4 marks)

2. (i) By expressing f(x) = $2\cos x - 3\sin x$ in the form $R\cos(x+\alpha)$, $0 < \alpha < \frac{\pi}{2}$, determine the maximum value of f(x) and the smallest positive value of x for which this maximum value occurs.

(6 marks)

(ii) Given that $0 < x < \frac{\pi}{2}$, prove that $\frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} = 2\operatorname{cosec} 2x$.

(4 marks)

(iii) (a) Find all solutions of the equation $\sin x = \frac{1}{2}$ in the range $-2\pi < x < 2\pi$.

(4 marks)

(b) Find all solutions of the equation $2\sin^2 2x - \sin 2x = 0$ in the range $-\pi < x < \pi$.

(6 marks)

3. (i) Find

(a) $\sum_{r=1}^{50}(2r-1)$

(b) $\sum_{r=1}^{\infty}\left(-\frac{1}{3}\right)^r$.

(6 marks)

(ii) In January 1 2000 I deposited £100 in a savings account which offers 15% interest per year, so that at the end of December 2000 I had £1.15×100 credited to me. In January 1 2001 I deposited another £100 into this account, so that at the end of December 2001 I had £1.15×(100 + 1.15×100) credited to me. I deposit £100 every January 1 and do not withdraw anything from the account. Show that at the end of December 2014 the amount that will be credited to me can be represented as a geometric series of 15 terms, and find the sum of this series to the nearest £.

(8 marks)

(iii) Write down the first four terms of the binomial expansion $(1+x)^{\frac{1}{2}}$. By substituting an appropriate value of x in this expansion find, to an accuracy of 3 decimal places, the value of $\sqrt{1.05}$.

(6 marks)

4. (i) Find the Cartesian equation of the curve described parametrically as

$$x = \tan t,\ \ y = 2\sec t,\ -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

(4 marks)

(ii) The Cartesian equation of a circle is $(x-5)^2 + y^2 = 9$. This circle can described parametrically as $x = a\cos t + b,\ y = a\sin t + c,\ 0 \le t < 2\pi$ and $a > 0$. Find the values of *a*, *b* and *c*.

(4 marks)

(iii) Let $z_1 = 2+3i$ and $z_2 = 3-2i$. Represent the complex numbers $u = \dfrac{z_1}{z_2}$ and $v = \dfrac{z_2}{z_1}$ in the form *a* + *bi*, where *a* and *b* are real numbers. Show that z_1 and z_2 lie on a circle centre $z = 0$ and that u and v lie on another a circle centre $z = 0$. State the equations of the two circles.

(8 marks)

(iv) Let $z_1 = 2+3i$ and $z_2 = 3-2i$. Describe the set $\left\{ z : \left| z - \dfrac{z_1}{z_2} \right| \le 1 \right\}$ in an Argand diagram.

(4 marks)

5. (i) The equation $x^4 - x^3 - 75 = 0$ has a root between 3 and 4. Use the Newton-Raphson method of approximating this root by taking $x_1 = 3.6$. Find x_4 to 4 places of decimals.

(6 marks)

(ii) Use the trapezium rule with 4 intervals to find an approximate value of the integral

$I = \pi \int_0^2 x^2 \sqrt{1+x^2}\, dx.$ Give your answer to 2 places of decimals.

(7 marks)

(iii) Given that *y* is a function of *x* that satisfies the differential equation $\dfrac{dy}{dx} = \dfrac{1-x}{y-1}$ subject to the condition that *y* = 0 when *x* =1. Show that the solution of this differential equation can be written as $(x-a)^2 + (y-a)^2 = b^2$. State the values of *a* and *b*.

(7 marks)

END OF QUESTION PAPER