The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS Spring Semester 2013–2014

Mathematics Core II 2 hours

Attempt all the questions. The allocation of marks is shown in brackets.

Section A

A1 Solve the following inequalities:

(i) $|3x - 3| < 9$; ***(2 marks)***

(ii) $|x^2 + x - 6| > 0$. ***(2 marks)***

A2 Evaluate the following limits:

(i) $\displaystyle\lim_{x\to 2} \frac{x^2 + x - 6}{x^2 - 3x + 2}$; ***(2 marks)***

(ii) $\displaystyle\lim_{x\to \pi/2} \frac{2x^2 - 3\pi x + \pi^2}{\cos x}$. ***(2 marks)***

A3 Give the equation of the tangent plane to the surface $z = 5x^2 + 3xy + y^2$ at the point $(1, 1, 9)$. ***(3 marks)***

A4 Let $f(x, y) = \dfrac{x - y}{\sqrt{x^2 - y^2}}$. Evaluate f at the point $(5, 4)$. Using partial derivatives, approximate the change δf when we move to the nearby point $(5.1, 3.9)$. ***(5 marks)***

A5 Let z be a function of u and v where $u = x - y$ and $v = 2x + 2y$.

(i) Show that

$$\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = 4\left(\frac{\partial z}{\partial v}\right)^2 - \left(\frac{\partial z}{\partial u}\right)^2.$$

(2 marks)

(ii) Assuming equality of mixed second-order partial derivatives, show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 2\frac{\partial^2 z}{\partial u^2} + 8\frac{\partial^2 z}{\partial v^2}.$$

(3 marks)

A6 Let R be the triangular region in the (x, y)-plane with vertices $(0, 0)$, $(2, 0)$ and $(0, 1)$. Calculate the integral

$$\int\int_R 2xy + 6x^2y \, dx \, dy.$$

(6 marks)

A7 What is the radius of convergence of $\sum_{n=1}^{\infty} \frac{(3n)!}{(n!)^3 5^n} z^n$? ***(3 marks)***

Section B

B1 Consider the following system of linear equations:

$$\begin{cases} x + 2y + 7z = 3, \\ -2x + 5y + 4z = 3, \\ -5x + 6y - 3z = 1. \end{cases}$$

By Gauss-Jordan elimination, put the relevant augmented matrix into a row-echelon form and solve the equations. ***(4 marks)***

B2 Let A, B be 2×2 matrices. For each statement below, decide whether it is true or false. Prove if it is true and give a counter-example if it is false.

(i) If $AB = 0$ and $A \neq 0$, then $B = 0$. ***(2 marks)***

(ii) If $AB = 0$ and $\det(A) \neq 0$, then $B = 0$. ***(2 marks)***

B3 By performing elementary row operations, show that

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z).$$

Hint:

$$x^3 - y^3 = (x-y)(x^2+xy+y^2).$$

(6 marks)

B4 Consider a family of curves on a plane which has *two* real parameters u and v

$$x = c\cosh u \cos v, \;\; y = c\sinh u \sin v,$$

where $c > 0$ is a constant and $0 \le v \le 2\pi$.

(i) By assuming that u is a constant, derive an equation of an ellipse

$$\frac{x^2}{c^2\cosh^2 u} + \frac{y^2}{c^2\sinh^2 u} = 1.$$

(2 marks)

(ii) By assuming that v is a constant, derive an equation of a hyperbola. ***(2 marks)***

(iii) Show that the two conics share their foci in common. ***(2 marks)***

(iv) Show that the two conics are perpendicular at each point of intersection, that is, their tangents are orthogonal to each other. ***(2 marks)***

Note: $\cosh x = \dfrac{e^x + e^{-x}}{2}$, $\sinh x = \dfrac{e^x - e^{-x}}{2}$.

B5 We prove that the points (p, q) which are the intersection of two orthogonal tangent lines from an ellipse lie on a circle.

(i) Let E be the ellipse given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $a, b > 0$. Let p and q be any real numbers. Find a quadratic equation in m which holds when a straight line through (p, q) with gradient m is a tangent to E. (You are encouraged to draw a picture.) ***(4 marks)***

(ii) By considering the two solutions for m above, show that the points (p, q) for which the two tangent lines to E that pass through (p, q) are orthogonal form a circle. ***(4 marks)***

End of Question Paper