The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2013–2014**

Vectors and Mechanics

2 **Hours**

Attempt all the questions. The allocation of marks is shown in brackets. The total number of marks available is 60.

1 Two points P and Q have position vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ respectively relative to an origin O, where

$$\boldsymbol{p} = 12\boldsymbol{i} + 5\boldsymbol{j}, \qquad \boldsymbol{q} = \boldsymbol{i} - 7\boldsymbol{j} + 2\boldsymbol{k}.$$

Find

(i) The position vector of the mid-point of PQ;

(ii) A unit vector parallel to the vector $\boldsymbol{p}$;

(iii) The vector $\overrightarrow{PQ}$;

(iv) The vector equation of the line PQ;

(v) The parametric vector equation of the plane containing the points O, P and Q. ***(5 marks)***

2 Find the values of x (if any) for which the following two lines intersect:

$$\begin{aligned} \boldsymbol{r}_1 &= \lambda\,(\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}) \\ \boldsymbol{r}_2 &= -3\boldsymbol{i} + 7\boldsymbol{j} + \mu\,(x\boldsymbol{i} - 15\boldsymbol{j} + 10\boldsymbol{k})\,. \end{aligned}$$

For what value of x are the two lines parallel? ***(4 marks)***

3 Given the vectors

$$\boldsymbol{u} = 3\boldsymbol{i} + 4\boldsymbol{j} + 5\boldsymbol{k}, \qquad \boldsymbol{v} = \boldsymbol{i}, \qquad \boldsymbol{w} = -3\boldsymbol{j} + z\boldsymbol{k},$$

find $(\boldsymbol{u} \times \boldsymbol{v})\,.\boldsymbol{w}$.

Find the value of z for which the three vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are coplanar.

(3 marks)

4 A force $\boldsymbol{F} = (-\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k})$ N acts on a particle of mass 0.5 kg.
At time $t = 0$ s the particle is at rest at the origin O.
Find

(i) The acceleration of the particle;

(ii) The velocity of the particle at time 4 s;

(iii) The position vector of the particle at time 4 s;

(iv) The power of the force at time $t = 4$ s;

(v) The impulse of the force between time $t = 0$ s and time $t = 4$ s.

(5 marks)

5 The points O, A, B and C lie on a straight line with $AB = 28$ m and $BC = 72$ m.

A particle moving along the straight line with constant acceleration starts from rest at O and passes through the points A, B and C.
At B its speed is 9 m s^{-1} and at C its speed is 15 m s^{-1}.
Find its speed at A. *(3 marks)*

6 A helicopter hovers at a height of 200 m above the ground.

A food parcel of mass 10 kg, attached to a parachute, is dropped at rest from the helicopter and hits the ground vertically below.

Use the value $g = 9.8$ m s^{-2} for the acceleration due to gravity.

Find the work done by air resistance if the speed of the food parcel is 20 m s^{-1} when it hits the ground. *(4 marks)*

7 A moon M of mass M_M is orbiting a planet P of mass M_P.

The position vector of the moon relative to the planet is given, at time t by:

$$\boldsymbol{r}(t) = R\left[\boldsymbol{i}\cos(\omega t) + \boldsymbol{j}\sin(\omega t)\right]$$

where R and ω are positive constants.

(i) Show that the moon moves on a circular orbit about the planet and find the radius of the orbit.

(ii) Find the velocity and acceleration of the moon relative to the planet.

(iii) If T is the period of the orbit, show that

$$\frac{R^3}{T^2} = \frac{GM_P}{4\pi^2}$$

where G is Newton's Universal Gravitational Constant.

(iv) Titan and Rhea are moons of the planet Saturn.

The orbit of Titan has a radius of 1.2×10^6 km and a period of 16 days.

The orbit of Rhea has a radius of 527000 km.

Find, in days correct to three significant figures, the period of the orbit of Rhea. ***(9 marks)***

8 A particle is projected from the origin with speed V at an angle θ above the horizontal.

If the horizontal and vertical displacements of the particle at time t are x and z respectively, write down expressions for x and z in terms of t, V, θ and the acceleration due to gravity g.

The particle just clears an obstacle of height h at a horizontal distance D from the origin.

Show that

$$\left[h + \frac{gD^2}{2V^2}\right] \leq \frac{V^2}{2g}.$$

(6 marks)

9 A particle of mass m is moving on a line of greatest slope of a rough plane inclined at an angle α to the horizontal, where $0 < \alpha < \pi/2$.

The coefficient of friction between the particle and the plane is $\frac{1}{2}\tan\alpha$.

The particle is attached to one end of an elastic spring of stiffness k, and the other end of the spring is attached to a fixed point O on the line of greatest slope of the plane.

The particle is released from rest with the spring unextended.

(i) Draw a clear diagram showing all the forces on the particle.

(ii) In the subsequent motion the extension of the spring is $x(t)$ with $x(t)$ increasing as the particle moves down the slope.
Show that

$$m\ddot{x} + kx = \frac{1}{2}mg\sin\alpha.$$

(iii) Hence find the extension of the spring $x(t)$ at time t.

(iv) Find the maximum extension of the spring in the subsequent motion.

(13 marks)

10 (i) Two lines have vector equations

$$\begin{aligned} \boldsymbol{r}_1 &= \boldsymbol{a} + \lambda\boldsymbol{c} \\ \boldsymbol{r}_2 &= \boldsymbol{a} + \mu\boldsymbol{b}. \end{aligned}$$

What are the conditions on the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ for these two lines to lie in a unique plane?

Assuming that these two lines lie in a unique plane, find, in the form $\boldsymbol{r}.\boldsymbol{n} = d$, the vector equation of that plane.

(ii) Now consider the two lines with vector equations

$$\begin{aligned} \boldsymbol{r}_3 &= \boldsymbol{a} + \lambda\boldsymbol{c} \\ \boldsymbol{r}_4 &= \boldsymbol{b} + \mu\boldsymbol{c}. \end{aligned}$$

What are the conditions on the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ for these two lines to lie in a unique plane?

Assuming that these two lines lie in a unique plane, find, in the form $\boldsymbol{r}.\boldsymbol{n} = d$, the vector equation of that plane. ***(8 marks)***

End of Question Paper