The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2013–2014**

Numbers and Groups **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) State the *Chinese Remainder Theorem.* ***(2 marks)***

(ii) Explaining your working, find the general solution to the following pair of simultaneous congruences:

$$x \equiv 11 \pmod{14},$$
$$x \equiv 9 \pmod{19}.$$

(6 marks)

(iii) Why does the following pair of simultaneous congruences have no solution?

$$x \equiv 9 \pmod{22},$$
$$x \equiv 14 \pmod{20}.$$

(2 marks)

2 (i) If n is a positive integer, we define the function $f_n : \mathbb{R} \to \mathbb{R}$ by the formula

$$f_n(x) = x^n.$$

For which positive integers n is this function injective? For which positive integers n is this function surjective? ***(2 marks)***

(ii) Show by induction on n that $n^5 - n$ is a multiple of 5 for all nonnegative integers n. ***(6 marks)***

(iii) State *Fermat's Little Theorem*, and explain how it shows that $n^4 - 1$ is a multiple of 5 for any nonnegative integer n which is not a multiple of 5. ***(2 marks)***

3 (i) Say what it means for a sequence $x_0, x_1, \ldots$ of real numbers to be a *Cauchy sequence.* ***(3 marks)***

(ii) Let a and k be positive integers. Using the fact $2^n \leq n^n$ for all positive integers n, or otherwise, show that

$$\frac{1}{(a+1)^{(a+1)}} + \frac{1}{(a+2)^{(a+2)}} + \cdots + \frac{1}{(a+k)^{(a+k)}} \leq \frac{1}{2^a}.$$

(4 marks)

(iii) Using the above, show that the sequence $x_0, x_1, \ldots$ defined by

$$x_n = \frac{1}{1^1} + \frac{1}{2^2} + \frac{1}{3^3} + \cdots + \frac{1}{n^n}$$

is a Cauchy sequence. ***(3 marks)***

4 (i) Consider the permutations

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 1 & 3 & 5 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 5 & 2 & 4 & 6 \end{pmatrix}.$$

(a) Show that β, $\alpha\beta$ and $\beta\alpha$ are cycles of length 4. ***(3 marks)***

(b) Express α as a product of disjoint cycles and determine its order. ***(2 marks)***

(c) Find a permutation $\rho \in S_6$ such that $\rho^2 = \beta^2$ but $\rho \neq \beta$. ***(1 mark)***

(ii) Let G be a group and let H be a subgroup of G.

(a) Let $g \in G$. Define what is meant by the left coset gH. ***(1 mark)***

(b) Show that the relation on G given by $a \sim b \iff b^{-1}a \in H$ is an equivalence relation.
(You may assume, without proof, that $e \in H$.) ***(3 marks)***

5 (i) State Lagrange's Theorem. ***(2 marks)***

(ii) Let A be a geometrical figure in $\mathbb{R}^2$ with centre at the origin. The set of symmetries of A is defined to be $S_A = \{f \in O_2 : f(A) = A\}$, where O_2 is the group of symmetries of the circle.

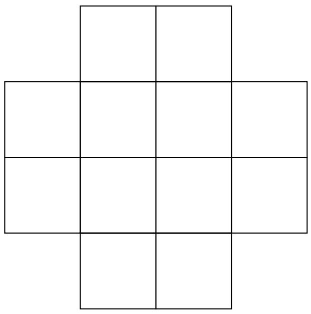
(a) By using the subgroup criterion, show that S_A is a subgroup of O_2. ***(3 marks)***

(b) Suppose that S_A has order 5. Let $f \in S_A$ be any non-identity element. State the order of f. Is it possible for S_A to contain a reflection? ***(3 marks)***

(c) Draw a geometrical figure in $\mathbb{R}^2$ with group of symmetries of order 5, and explain how to extend this to a figure with group of symmetries of order p, where $p > 2$ is any prime number. ***(2 marks)***

6 (i) Let G be a finite group acting on a non-empty finite set X and let n be the number of orbits. Write down a formula relating n, the order of G and the sizes of the fixed sets, $\text{fix}(g)$ (for $g \in G$). ***(1 mark)***

(ii) A stained-glass tile is to be formed from small coloured glass squares stuck together in the shape below.



Find the number of essentially different colourings of the tile using 8 green and 4 blue squares, making your method clear. ***(9 marks)***

End of Question Paper