Data provided: Formula Sheet

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2013–2014**

MAS140 Mathematics (Chemical)
MAS151 Civil Engineering Mathematics
MAS152 Essential Mathematical Skills And Techniques

3 hours

*Attempt **ALL** questions.*

Each question in Section A carries 3 marks, each question in Section B carries 8 marks.

Section A

A1 Let $f(x) = \dfrac{9}{x+3}$ with domain $x \geq 0$. Sketch the curve $y = f(x)$.

What is the range of $f(x)$? Find $f^{-1}(x)$.

A2 Use the binomial theorem to evaluate $\lim\limits_{x\to\infty} \left[\sqrt{x^2 - 2x - 9} - x\right]$

A3 If $f(x, y) = \dfrac{x^2 \sin 2y - 3y \cos x}{x - 1}$, find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

A4 Evaluate $\lim\limits_{x\to\infty} \left[(x^2 + 2x + 3)e^{-x}\right]$ using l'Hôpital's rule.

A5 Find all the complex numbers z for which $z^3 = -1 + i$.

A6 Find the angle between the vectors $\mathbf{a} = (1, 4, -1)$ and $\mathbf{b} = (-1, 1, 2)$.

A7 Use integration by parts to evaluate the definite integral

$$\int_0^{\pi} x^2 \sin(2x) dx$$

A8 Find the indefinite integral

$$\int \frac{x+1}{x^2+1} dx$$

A9 Let

$$A(x) = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}.$$

Find the determinant of $A(x)$, and show that $A(x)A(x) = A(2x)$.

A10 Find the solution $y(x)$ of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y(x) = 0$$

with boundary conditions $y(0) = 0$ and $y(1) = 1$.

A11 Solve the following system of linear equations

$$\begin{array}{rrrcl} x & +y & -2z & = & 3 \\ 2x & +y & +z & = & 4 \\ 3x & & +z & = & 3 \end{array}$$

for x, y and z.

A12 Show that $y = x\ln(\ln x)$ is a solution of the differential equation

$$\frac{dy}{dx} = \exp\left(-\frac{y}{x}\right) + \frac{y}{x}.$$

Section B

B1 Find all the minima and maxima of the function

$$f(x) = \frac{x^2 + 4x + 10}{x^2 + 4}.$$

Find the range of $f(x)$ and sketch the curve $y = f(x)$.

B2 Find the first four non-zero terms in the Maclaurin expansion of $f(x) = e^{-x} \cos x$.

B3 From the definition of $\cosh x$, show that

$$\cosh 4x = 8 \cosh^4 x - 8 \cosh^2 x + 1.$$

B4 The position vector $\mathbf{r}(t)$ of a particle is given by

$$\mathbf{r} = (2 \cos t \cos 3t,\ 2 \cos t \sin 3t,\ \sin t),\ t \geq 0.$$

(a) Find the minimum and maximum distances of the particle from the origin, and the corresponding position vectors.

(b) Find the velocity vector of the particle. Show that the maximum speed of the particle is attained at the same time as the maximum distance from the origin, and find this speed.

(c) Describe the trajectory of the particle.

B5 Let

$$f(x) = \frac{x}{(x+1)(x^2+4)}$$

(a) Use the method of partial fractions to show that

$$f(x) = \frac{x+4}{5(x^2+4)} - \frac{1}{5(x+1)}$$

(b) Show that

$$\int_0^2 f(x)dx = \frac{1}{10}\left(\pi - \ln(9/2)\right).$$

(c) Evaluate $\int_0^\infty f(x)dx$.

B6 Let

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 3 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}$$

(a) Find AB. Show that

$$BA = \begin{bmatrix} 0 & -3 & 9 \\ 1 & -1 & 5 \\ -1 & -1 & 1 \end{bmatrix}$$

(b) Find the determinants $|BA|$ and $|AB|$.

(c) Find the matrix inverse of AB. Why can't BA be inverted?

B7 Let

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 2 \\ 1 & 2 & -1 \end{bmatrix}, \qquad b = \begin{bmatrix} -4 \\ 5 \\ 2 \end{bmatrix}$$

(a) By multiplying A and b, show that

$$Ab = 2b.$$

Hence b is an eigenvector of A. State the corresponding eigenvalue.

(b) Find two more eigenvalues and corresponding eigenvectors of A.

B8 Consider the following differential equation,

$$y'' + 2y' + (1 + a^2)y = 0,$$

where a is a positive constant.

(a) Find a solution $y(x)$ satisfying initial conditions $y(0) = 0$ and $y'(0) = a$.

(b) Show that $y(x)$ has stationary points where $x = \dfrac{1}{a}\tan^{-1}(a)$.

(c) For the case $a = 1$, find the maximum value of $y(x)$ in the domain $x > 0$.

End of Question Paper