The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2013–2014**

MAS156 Mathematics (Electrical and Aerospace) **2 hours 30 minutes**

Attempt ALL questions
Each question in Section A carries 2 or 3 marks and each question in Section B carries 8 marks

Section A

A1 Give an example of a function of period 5π. Give an example of a function of period 6π which does *not* have fundamental period 6π. ***(2 marks)***

A2 Calculate the derivative with respect to x of $f(x) = \dfrac{\tan x \sin x}{x}$. ***(3 marks)***

A3 Calculate $(0,0,1) \cdot \big((4,5,6) \times (1,0,0)\big)$. ***(3 marks)***

A4 What is the imaginary part of $\dfrac{2+3j}{4-5j}$? ***(3 marks)***

A5 Evaluate the indefinite integral $\displaystyle\int \frac{x}{1+2x}\,\mathrm{d}x$. ***(3 marks)***

A6 Find the general solution of the differential equation $\dfrac{dy}{dt} + 3y = e^{-t}$. *(3 marks)*

A7 Find the inverse Laplace transform of the function $\dfrac{3s+7}{s^2+4s+8}$. *(3 marks)*

A8 Find $\displaystyle\lim_{x\to\infty} \frac{5x^2-3x+2}{2x^2-3}$. *(3 marks)*

A9 Find the inverse of the matrix $A = \begin{pmatrix} 1 & -1 \\ -6 & 7 \end{pmatrix}$. *(3 marks)*

A10 Find the eigenvalues of the matrix $M = \begin{pmatrix} -3 & 4 \\ 2 & -1 \end{pmatrix}$. *(3 marks)*

Section B

B1 Use De Moivre's Theorem to show that

$$\cos(4\theta) = 8\cos^4\theta - 8\cos^2\theta + 1.$$

If $\cos(4\theta) = 0$ then find the possible values of $\cos^2\theta$. Use that (and other knowledge of the cos function) to find $\cos(\pi/8)$.

B2 Find the angles between each of the following pairs of vectors:

(i) $(1,2,3)$ and $(-5,1,1)$;

(ii) $(1,1,1)$ and $(-2,-2,-2)$;

(iii) $(1,2,3)$ and $(1,1,1)$.

B3 Find the maximum and minimum values attained by the function

$$f(x) = 8x^3 - 20x^2 + 6x + 9$$

on the interval $[-2,2]$.

B4 Evaluate the improper integral

$$\int_0^\infty x^2 e^{-2x}\ \mathrm{d}x.$$

B5 Find the solution of the differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 5y = 2\cos 3t,$$

given that $y(0) = 1$ and $\dfrac{dy}{dt}(0) = 0$.

B6 Sketch the function

$$f(x) = \frac{1}{8 + x^3}, \quad x \geq 0.$$

Devise the Newton-Raphson iterative scheme to solve the equation

$$f(x) = \frac{x}{10}, \quad x \geq 0.$$

Show from your sketch that the equation has a single solution and that an appropriate first approximation to the solution is $x_0 = 1$. Using x_0 as a first approximation, find the solution correct to four decimal places.

B7 Given the matrices

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 2 & 0 \\ 3 & -2 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} -5 & 1 \\ 2 & 0 \\ 1 & 1 \end{pmatrix},$$

state which of the following exist, giving brief reasons for your decisions.

(i) $A + B$, (ii) $B + C^T$, (iii) BC, (iv) $B^T A$.

Calculate the values of those that do exist.

End of Question Paper

Formula Sheet for MAS156

Trigonometry
$\cos^2\theta = (1+\cos 2\theta)/2$
$\sin^2\theta = (1-\cos 2\theta)/2$
$\sin 2\theta = 2\sin\theta\cos\theta$

Hyperbolic Functions
$\cosh^2\theta = (1+\cosh 2\theta)/2$
$\sinh^2\theta = -(1-\cosh 2\theta)/2$
$\sinh 2\theta = 2\sinh\theta\cosh\theta$

Binomial theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \binom{n}{r}x^r + \ldots$$

where $\binom{n}{r} = \dfrac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$.
If n is a positive integer then the series terminates and the result is true for all x, otherwise, the series is infinite and only converges for $|x| < 1$.

Function	Derivative
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\sin^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{a^2-x^2}}$
$\cos^{-1}\left(\frac{x}{a}\right)$	$\frac{-1}{\sqrt{a^2-x^2}}$
$\tan^{-1}\left(\frac{x}{a}\right)$	$\frac{a}{a^2+x^2}$
$\sinh^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{x^2+a^2}}$
$\cosh^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{x^2-a^2}}$
$\tanh^{-1}\left(\frac{x}{a}\right)$	$\frac{a}{a^2-x^2}$

Integration-by-Parts

$$\int uv'\,dx = uv - \int u'v\,dx$$

Substitution for a Rational Function of $\sin x$ **and** $\cos x$

If $t = \tan\left(\frac{x}{2}\right)$ then $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$ and $\frac{dx}{dt} = \frac{2}{1+t^2}$.

Taylor expansion of $f(x)$ **about** $x = a$

$$f(a) + (x-a)f^{(1)}(a) + \frac{(x-a)^2}{2!}f^{(2)}(a) + \ldots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \ldots$$

Newton-Raphson formula for the root of $f(x) = 0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Table of Laplace transforms

$f(t)$	$F(s) = \mathcal{L}(f(t))$
t^n	$\frac{n!}{s^{n+1}}$ $(n = 0, 1, 2, \ldots)$
e^{at}	$\frac{1}{s-a}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sinh \omega t$	$\frac{\omega}{s^2-\omega^2}$
$\cosh \omega t$	$\frac{s}{s^2-\omega^2}$
$e^{at}f(t)$	$F(s-a)$ (shift theorem)
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$