Data provided: Formula sheet

MAS157

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2013–2014**

Mathematics for Chemists **2 hours**

The marks awarded to each question or section of question are shown in italics.

1 (a) Showing your working clearly, find the coefficient of x^2 in the expansion of $(1+x)^{41}$. ***(3 marks)***

(b) Use the first four non-zero terms in the binomial theorem to find an approximation to $\sqrt[3]{1.03}$, keeping eight decimal places in your answer. ***(5 marks)***

(c) Use the binomial theorem to evaluate

$$\lim_{x\to\infty}\left[x+2-\sqrt{x^2+4x+8}\right].$$ ***(5 marks)***

2 (a) Vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are given by

$$\mathbf{a}=(-1,4,2),\qquad \mathbf{b}=(3,-2,1),\qquad \mathbf{c}=(2,3,-1).$$

Find $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ and $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$. ***(7 marks)***

(b) A line L_1 passes through the points $(1,-1,2)$ and $(3,1,1)$, and a line L_2 passes through the points $(-2,0,1)$ and $(14,8,-2)$.

Show that the lines intersect, and find the coordinates of the point of intersection. ***(9 marks)***

3 (a) Prove, from the definitions of $\sinh x$ and $\cosh x$, the identity

$$\sinh x\, \cosh y + \cosh x\, \sinh y = \sinh(x+y).$$ ***(5 marks)***

(b) Let $y = \sinh^{-1} x$ and $z = e^y$.
Show that

$$z^2 - 2xz - 1 = 0,$$

and hence show that

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right).$$ ***(8 marks)***

4 (a) If $y = \ln(\cosh x)$, show that

$$\frac{dy}{dx} = \tanh x.$$ ***(3 marks)***

(b) If $y = \dfrac{\sinh x\, \cosh 2x}{\cosh^2 x + \sinh^2 x}$ find $\dfrac{dy}{dx}$. ***(3 marks)***

5 (a) By making the substitution $x = a\cosh\theta$, where a is a positive constant, show that

$$\int_a^{2a} \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1} 2.$$ ***(8 marks)***

(b) Evaluate

$$\int \frac{4x^2-13x+13}{(x+1)(x^2-4x-5)}\, dx.$$ ***(10 marks)***

6 Find the Maclaurin series for $\cosh 2x$, as far as the term in x^4. ***(6 marks)***

7 (a) Complex numbers z_1 and z_2 are defined by

$$z_1 = 2+i, \qquad z_2 = -1+2i.$$

Find, in the form $a+bi$ where a and b are real, $\dfrac{z_1}{2z_1+z_2}$. ***(5 marks)***

(b) Use de Moivre's theorem to show that

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta,$$

and to express $\sin 3\theta$ in terms of $\sin\theta$. ***(10 marks)***

8 A set of linear equations can be written as

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ -6 \\ 4 \end{pmatrix},$$

where

$$A = \begin{pmatrix} -1 & 2 & 3 \\ 1 & -1 & -2 \\ 2 & 1 & 3 \end{pmatrix}.$$

Find the inverse, A^{-1}, of A, and use it to find the values of x, y and z which satisfy the equations. ***(13 marks)***

End of Question Paper

Formula Sheet for MAS153/MAS157/MAS159 Examination

These results may be quoted without proof, unless proofs are asked for in the question.

Trigonometry

For any angles A and B

$$\sin^2 A + \cos^2 A = 1$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

Coordinate Geometry

The acute angle α between lines with gradients m_1 and m_2 satisfies

$$\tan \alpha = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right| \qquad (m_1 m_2 \neq -1)$$

while the lines are perpendicular if $m_1 m_2 = -1$.
The equation of a circle centre (x_0, y_0) and radius a is $(x - x_0)^2 + (y - y_0)^2 = a^2$.

Hyperbolic Functions

$$\cosh^2 x - \sinh^2 x = 1$$

$$\mathrm{sech}^2 x + \tanh^2 x = 1$$

$$\cosh^2 x + \sinh^2 x = \cosh 2x$$

$$2 \sinh x \cosh x = \sinh 2x$$

$$\cosh^2 x = (1 + \cosh 2x)/2$$

$$\sinh^2 x = -(1 - \cosh 2x)/2$$

Differentiation

Function (y)	**Derivative** (dy/dx)
x^n	nx^{n-1}
$\sin ax$	$a\cos ax$
$\cos ax$	$-a\sin ax$
$\tan ax$	$a\sec^2 ax$
e^{ax}	ae^{ax}
$\ln(ax)$	$\dfrac{1}{x}$
$\ln f(x)$	$\dfrac{f'(x)}{f(x)}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\text{sech}^2 x$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$
$\sinh^{-1} x$	$\dfrac{1}{\sqrt{x^2+1}}$
$\cosh^{-1} x$	$\dfrac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\dfrac{1}{1-x^2}$

NB. It is assumed that x takes only those values for which the functions are defined.

For u and v functions of x, and with $u' = \frac{du}{dx}$, $v' = \frac{dv}{dx}$,

$$\frac{d}{dx}(uv) = uv' + vu',$$

while

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}.$$

For $y = y(t)$, $t = t(x)$,

$$\frac{dy}{dx} = \frac{dy}{dt}\,\frac{dt}{dx}.$$

Integration

In the following table the constants of integration have been omitted.

Function $f(x)$	**Integral** $\int f(x)\,dx$	
x^n	$\dfrac{x^{n+1}}{n+1}$	$n \neq -1$
ae^{ax}	e^{ax}	
$\dfrac{1}{x}$	$\ln\lvert x\rvert$	
$a\sin ax$	$-\cos ax$	
$a\cos ax$	$\sin ax$	
$a\tan ax$	$\ln\lvert\sec ax\rvert$	
$\dfrac{1}{a^2+x^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$	
$\dfrac{1}{a^2-x^2}$	$\dfrac{1}{a}\tanh^{-1}\left(\dfrac{x}{a}\right)$	
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)$	
$\dfrac{1}{\sqrt{x^2+a^2}}$	$\sinh^{-1}\left(\dfrac{x}{a}\right)$	
$\dfrac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\left(\dfrac{x}{a}\right)$	
$\dfrac{f'(x)}{f(x)}$	$\ln\lvert f(x)\rvert$	

Integration by parts

$$\int uV\,dx = (\text{integral of } V) \times u - \int (\text{integral of } V) \times \frac{du}{dx}\,dx$$

or

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx.$$

Series

Binomial Theorem: $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \cdots + \dbinom{n}{r}x^r + \cdots$

$$\text{where } \binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

If n is a positive integer, the series terminates and is convergent for all x.

If n is not a positive integer, the series is infinite and converges for $|x| < 1$.

Taylor expansion of $f(x)$ about $x = a$ is

$$f(a) + (x-a)f^{(1)}(a) + \frac{(x-a)^2}{2!}f^{(2)}(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \cdots$$

Maclaurin expansion of $f(x)$ is

$$f(0) + xf^{(1)}(0) + \frac{x^2}{2!}f^{(2)}(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \cdots$$

Alternating Series Test

The series $a_1 - a_2 + a_3 - a_4 + \cdots$, where $a_1, a_2, a_3, a_4, \ldots$ are all positive, converges if $a_1 > a_2 > a_3 > \cdots$ and $a_n \to 0$ as $n \to \infty$.

Ratio Test

If the series $a_1 + a_2 + a_3 + a_4 + \cdots$ satisfies

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lambda,$$

then

1. if $\lambda > 1$, the series diverges,
2. if $\lambda < 1$, the series converges.

Vectors

If vectors $\mathbf{a}$ and $\mathbf{b}$ are given in Cartesian component form by $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, then

the scalar product $\mathbf{a} \cdot \mathbf{b}$ is given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

and the vector product $\mathbf{a} \times \mathbf{b}$ is given by

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1).$$

If a plane passes through a point with position vector $\mathbf{a}$, and is normal to the vector $\mathbf{n}$, then the equation of the plane is

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n},$$

where $\mathbf{r} = (x, y, z)$.