Data provided: Formula sheet

MAS153/MAS159

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2013–2014**

Mathematics (Materials)
Mathematics For Chemists **3 hours**

All questions are compulsory. The marks awarded to each question or section of question are shown in italics.

1 Solve the equation

$$3\tan^2 x - 2\sec^2 x = 1$$

for $0 < x < \pi$. ***(5 marks)***

2 Factorise $x^2 + 8x + 15$. ***(2 marks)***

3 Solve $-0.1x^2 + 1.1x - 2.8 = 0$ for x. ***(4 marks)***

4 (i) A line with gradient $\frac{2}{3}$ passes through the point P(5,7). Find the x-coordinate of the point Q $(x, 13)$ on the line. ***(3 marks)***

(ii) Show that the line through the points A(8,10) and B(4,8) is perpendicular to line through the points C (6,0) and D(0,12). ***(3 marks)***

(iii) Show that

$$x^2 + y^2 - x + \frac{6}{5}y = \frac{39}{100}$$

is an equation of a circle. Find the centre and radius of the circle. ***(5 marks)***

(iv) Find the equations of the tangent and the normal to the circle

$$x^2 + y^2 - 5x + 2y = -3$$

at the point (2,-3). ***(8 marks)***

5 Solve

$$\log_2(x+1) + \log_2(x) = 1$$

(3 marks)

6 Find the inverse function, $f^{-1}(x)$, of the one-to-one function $f(x) = x^3 - 2$ valid for all real x. State the domain and range of $f^{-1}(x)$. ***(5 marks)***

7 A car makes a round trip journey from point A to point B, these points being separated by 40 km. The car drives at 40 km per hour from point A to point B, but returns at 60 km per hour. Find the average speed of the car. Show that the average speed is given by the harmonic mean of the speeds in each direction.

(5 marks)

8 If $y = (1 + x^2 + 4x^3)/\sin x$, then find dy/dx. ***(4 marks)***

9 Evaluate

$$\int \frac{1}{(x+1)^2} dx$$

(3 marks)

10 (i) Showing your working clearly, find the coefficient of x^2 in the expansion of $(1+x)^{41}$. ***(2 marks)***

(ii) Use the first four non-zero terms in the binomial theorem to find an approximation to $\sqrt[3]{1.03}$, keeping eight decimal places in your answer.

(3 marks)

(iii) Use the binomial theorem to evaluate

$$\lim_{x \to \infty} \left[x + 2 - \sqrt{x^2 + 4x + 8} \right].$$ ***(3 marks)***

11 A line L_1 passes through the points $(1, -1, 2)$ and $(3, 1, 1)$, and a line L_2 passes through the points $(-2, 0, 1)$ and $(14, 8, -2)$.

Show that the lines intersect, and find the coordinates of the point of intersection.

(6 marks)

12 Let $y = \sinh^{-1} x$ and $z = e^y$.

Show that

$$z^2 - 2xz - 1 = 0,$$

and hence show that

$$\sinh^{-1} x = \ln \left(x + \sqrt{x^2 + 1} \right).$$ ***(5 marks)***

13 (i) If $y = \ln(\cosh x)$, show that

$$\frac{dy}{dx} = \tanh x.$$ *(2 marks)*

(ii) If $y = \dfrac{\sinh x \, \cosh 2x}{\cosh^2 x + \sinh^2 x}$ find $\dfrac{dy}{dx}$. *(2 marks)*

14 Evaluate

$$\int \frac{4x^2 - 13x + 13}{(x+1)(x^2 - 4x - 5)} \, dx.$$ *(7 marks)*

15 Find the Maclaurin series for $\cosh 2x$, as far as the term in x^4. *(4 marks)*

16 Use de Moivre's theorem to show that

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta,$$

and to express $\sin 3\theta$ in terms of $\sin\theta$. *(7 marks)*

17 A set of linear equations can be written as

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ -6 \\ 4 \end{pmatrix},$$

where

$$A = \begin{pmatrix} -1 & 2 & 3 \\ 1 & -1 & -2 \\ 2 & 1 & 3 \end{pmatrix}.$$

Find the inverse, A^{-1}, of A, and use it to find the values of x, y and z which satisfy the equations. *(9 marks)*

End of Question Paper

Formula Sheet for MAS153/MAS157/MAS159 Examination

These results may be quoted without proof, unless proofs are asked for in the question.

Trigonometry

For any angles A and B

$\sin^2 A + \cos^2 A = 1$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$\sin 2A = 2 \sin A \cos A$

$\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

Coordinate Geometry

The acute angle α between lines with gradients m_1 and m_2 satisfies

$$\tan \alpha = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right| \qquad (m_1 m_2 \neq -1)$$

while the lines are perpendicular if $m_1 m_2 = -1$.
The equation of a circle centre (x_0, y_0) and radius a is $(x - x_0)^2 + (y - y_0)^2 = a^2$.

Hyperbolic Functions

$\cosh^2 x - \sinh^2 x = 1$

$\text{sech}^2 x + \tanh^2 x = 1$

$\cosh^2 x + \sinh^2 x = \cosh 2x$

$2 \sinh x \cosh x = \sinh 2x$

$\cosh^2 x = (1 + \cosh 2x)/2$

$\sinh^2 x = -(1 - \cosh 2x)/2$

Differentiation

Function (y)	**Derivative** (dy/dx)
x^n	nx^{n-1}
$\sin ax$	$a\cos ax$
$\cos ax$	$-a\sin ax$
$\tan ax$	$a\sec^2 ax$
e^{ax}	ae^{ax}
$\ln(ax)$	$\frac{1}{x}$
$\ln f(x)$	$\frac{f'(x)}{f(x)}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\text{sech}^2 x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$

NB. It is assumed that x takes only those values for which the functions are defined.

For u and v functions of x, and with $u' = \frac{du}{dx}$, $v' = \frac{dv}{dx}$,

$$\frac{d}{dx}(uv) = uv' + vu',$$

while

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}.$$

For $y = y(t)$, $t = t(x)$,

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}.$$

Integration

In the following table the constants of integration have been omitted.

Function $f(x)$	**Integral** $\int f(x)\,dx$	
x^n	$\frac{x^{n+1}}{n+1}$	$n \neq -1$
ae^{ax}	e^{ax}	
$\frac{1}{x}$	$\ln\|x\|$	
$a\sin ax$	$-\cos ax$	
$a\cos ax$	$\sin ax$	
$a\tan ax$	$\ln\|\sec ax\|$	
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$	
$\frac{1}{a^2-x^2}$	$\frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right)$	
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right)$	
$\frac{1}{\sqrt{x^2+a^2}}$	$\sinh^{-1}\left(\frac{x}{a}\right)$	
$\frac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\left(\frac{x}{a}\right)$	
$\frac{f'(x)}{f(x)}$	$\ln\|f(x)\|$	

Integration by parts

$$\int uV\,dx = (\text{integral of } V) \times u - \int (\text{integral of } V) \times \frac{du}{dx}\,dx$$

or

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx.$$

Series

Binomial Theorem: $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + \binom{n}{r}x^r + \cdots$

$$\text{where } \binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

If n is a positive integer, the series terminates and is convergent for all x.

If n is not a positive integer, the series is infinite and converges for $|x| < 1$.

Taylor expansion of $f(x)$ about $x = a$ is

$$f(a) + (x-a)f^{(1)}(a) + \frac{(x-a)^2}{2!}f^{(2)}(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \cdots$$

Maclaurin expansion of $f(x)$ is

$$f(0) + xf^{(1)}(0) + \frac{x^2}{2!}f^{(2)}(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \cdots$$

Alternating Series Test

The series $a_1 - a_2 + a_3 - a_4 + \cdots$, where $a_1, a_2, a_3, a_4, \ldots$ are all positive, converges if $a_1 > a_2 > a_3 > \cdots$ and $a_n \to 0$ as $n \to \infty$.

Ratio Test

If the series $a_1 + a_2 + a_3 + a_4 + \cdots$ satisfies

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lambda,$$

then

1. if $\lambda > 1$, the series diverges,

2. if $\lambda < 1$, the series converges.

Vectors

If vectors $\mathbf{a}$ and $\mathbf{b}$ are given in Cartesian component form by $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, then

the scalar product $\mathbf{a} \cdot \mathbf{b}$ is given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

and the vector product $\mathbf{a} \times \mathbf{b}$ is given by

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1).$$

If a plane passes through a point with position vector $\mathbf{a}$, and is normal to the vector $\mathbf{n}$, then the equation of the plane is

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n},$$

where $\mathbf{r} = (x, y, z)$.