Data provided: Formula sheet

MAS165

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS Spring Semester 2013–2014

MAS165: Mathematics for Physicists 2 hours

You should attempt ALL questions of this exam.

Section A

A1 (i) Let $A = (1, -2, 0)$, $B = (3, 1, 4)$ and $C = (0, -1, 2)$. Find the equation of the plane passing through A, B and C. ***(5 marks)***

(ii) Find the vector equation of the line through A and B. ***(3 marks)***

(iii) Find the distance from the plane in part (i) to the point $D = (0, 0, 1)$. ***(5 marks)***

A2 (i) Let $\mathbf{V}(t) = (t, t^2, t^3)$ and $\mathbf{W}(t) = (e^t, e^{2t}, e^{3t})$. Find

$$\frac{d}{dt}(\mathbf{V} \cdot \mathbf{W}) \quad \text{and} \quad \frac{d}{dt}(\mathbf{V} \times \mathbf{W}).$$

(5 marks)

(ii) Let $|\mathbf{r}(t)|$ be constant. Show that $\mathbf{r}(t)$ and $d\mathbf{r}/dt$ are perpendicular. ***(3 marks)***

A3 Gauss' divergence theorem may be written:

$$\int\int\int_R \nabla \cdot \boldsymbol{F}\, dV = \int\int_S \boldsymbol{F} \cdot \boldsymbol{n}\, dA$$

Indicate whether the following statements about Gauss' divergence theorem, as expressed here, are true or false.

(i) The term $\nabla \cdot \boldsymbol{F}$ is the curl of the vector field $\boldsymbol{F}$.

(ii) The volume R is bounded by the surface S.

(iii) $\boldsymbol{n}$ is a unit vector parallel to the surface S.

(iv) $\int\int_S dA$ is a surface integral, over the surface S.

(4 marks)

Section B

B1 (i) Let $\phi(x, y, z) = x^3 + 2xy^2z - y^3$. Calculate $\nabla\phi$ and $\nabla \times (\nabla\phi)$. ***(6 marks)***

(ii) Let $\boldsymbol{F} = \nabla\phi$. Let C be the straight line path from $(0, 0, 0)$ to $(2, 4, 8)$. Calculate the line integral

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$$

(8 marks)

(iii) Let D be the path $\boldsymbol{r}(t) = (t, t^3, t^3)$, where $0 \leq t \leq 2$. Calculate the line integral

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$$

(8 marks)

(iv) Explain why the answers in (ii) and (iii) are equal. ***(3 marks)***

B2 (i) Let

$$f(x, y) = e^{3(x+ct)} - (x - ct)^3.$$

where c is a constant.

Verify that f satisfies the *wave equation*

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}$$

(5 marks)

(ii) Let $\phi(x, y, z) = x^3 - y^2 z + 2yz^2 - z^3$. Find the directional derivative of ϕ at $(1, 0, 1)$ in the direction of the vector $(3, 4, 0)$. ***(5 marks)***

(iii) Sketch the circular section R where $x^2 + y^2 \leq a^2$ and $y \geq 0$. Evaluate the integrals

$$\int\int_R x^2 + y^2 \; dx \; dy \qquad \int\int_R x^2 y \; dx \; dy$$

by transforming to plane polar coordinates. ***(15 marks)***

B3 (i) Let

$$V(\boldsymbol{r}) = \frac{k}{|\boldsymbol{r}|}$$

where k is constant

Using spherical polar coordinates or otherwise, find ∇V in the simplest possible terms, and show that $\nabla^2 V = 0$. ***(10 marks)***

(ii) Find $curl \, \nabla V$, where V is as above. ***(3 marks)***

(iii) Using Gauss' divergence theorem or otherwise, calculate

$$\int\int_S \boldsymbol{F} \cdot \hat{\boldsymbol{n}} \; dA$$

where $\mathbf{F}$ is the vector field $(x^4, x^2 y z^2, \sin^4(xy))$ and S is the surface of the cuboid described by the inequalities $-1 \leq x \leq 1$, $-2 \leq y \leq 2$ and $-1 \leq z \leq 1$. ***(12 marks)***

End of Question Paper

Mathematical Formulae:

Spherical Polar Coordinates:

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta,$$
$$dV = r^2\sin\theta dr d\theta d\phi \quad \text{(Element of volume)}$$

In the following $\mathbf{F} = F_1\hat{\mathbf{r}} + F_2\hat{\boldsymbol{\theta}} + F_3\hat{\boldsymbol{\phi}}$ (note that $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ are unit vectors):

$$\boldsymbol{\nabla}\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2F_1\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta F_2\right) + \frac{1}{r\sin\theta}\frac{\partial F_3}{\partial\phi}$$

and

$$\nabla\times\mathbf{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ F_1 & rF_2 & r\sin\theta F_3 \end{vmatrix}.$$

Let f be a scalar function, then the gradient is given by

$$\text{grad} f = \boldsymbol{\nabla} f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\hat{\boldsymbol{\phi}}.$$

Plane Polar Coordinates:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad dA = dxdy = rdrd\theta$$

Vector Calculus:

$$\begin{aligned}
\boldsymbol{\nabla}\phi &= \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k} \\
\boldsymbol{\nabla}\cdot\mathbf{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\
\boldsymbol{\nabla}\times\mathbf{A} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{i} - \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\mathbf{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{k} \\
\boldsymbol{\nabla}^2\phi &= \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\phi) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \\
\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\phi) &= \mathbf{0} \\
\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) &= 0 \\
\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}) &= \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}
\end{aligned}$$

Vectors:

$$
\begin{aligned}
\mathbf{A}\cdot\mathbf{B} &= A_1B_1 + A_2B_2 + A_3B_3 \\
\mathbf{A}\times\mathbf{B} &= (A_2B_3 - A_3B_2)\mathbf{i} - (A_1B_3 - A_3B_1)\mathbf{j} + (A_1B_2 - A_2B_1)\mathbf{k} \\
\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) &= (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C} \\
\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})
\end{aligned}
$$

Trigonometry:

$$
\begin{aligned}
\sin(\phi \pm \theta) &= \sin\phi\cos\theta \pm \cos\phi\sin\theta \\
\cos(\phi \pm \theta) &= \cos\phi\cos\theta \mp \sin\phi\sin\theta \\
\tan(\theta \pm \phi) &= \frac{\tan\phi \pm \tan\theta}{1 \mp \tan\phi\tan\theta} \\
\sin(2\phi) &= 2\sin\phi\cos\phi \\
\cos(2\phi) &= 2\cos^2\phi - 1 = 1 - 2\sin^2\phi \\
\sin\phi + \sin\theta &= 2\sin\left(\frac{\phi+\theta}{2}\right)\cos\left(\frac{\phi-\theta}{2}\right) \\
\sin\phi - \sin\theta &= 2\cos\left(\frac{\phi+\theta}{2}\right)\sin\left(\frac{\phi-\theta}{2}\right) \\
\cos\phi + \cos\theta &= 2\cos\left(\frac{\phi+\theta}{2}\right)\cos\left(\frac{\phi-\theta}{2}\right) \\
\cos\phi - \cos\theta &= 2\sin\left(\frac{\phi+\theta}{2}\right)\sin\left(\frac{\phi-\theta}{2}\right)
\end{aligned}
$$