The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2013–2014

Mathematics (Computational and Numerical Methods) — 2 hours

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) The function $f(x) = x^3 - x - 1$ has a root near $x = 1.5$. Starting with $x_0 = 1.5$, use the Newton-Raphson method to compute the next three root approximations x_1, x_2 and x_3 accurate to four decimal places.

(8 marks)

(ii) Show that the function

$$f(x) = 3x - \cos x - 1 \qquad (1)$$

has a root in the interval $(0.55, 0.75)$. What is the reduced interval after performing five iterations of the bisection method? Give your answer accurate to four decimal places. When applying the bisection method for finding the root of equation (1), the error ϵ is bounded using the formula

$$\frac{1}{2^n}(b-a) \leq \epsilon,$$

where (a, b) is the initial interval. To obtain the root value with $\epsilon \leq 10^{-5}$ estimate the minimum number of iterations, n, required using the same starting interval.

(17 marks)

2 (i) Starting with the initial column vector $\mathrm{x} = [2, -1, 1]^T$ use the Gauss-Seidel method to compute four successive approximations to the solution of the following system,

$$\begin{aligned} 4x_1 - 2x_2 - x_3 &= 8 \\ 3x_1 - 5x_2 + x_3 &= 10 \\ x_1 + x_2 + 3x_3 &= -6. \end{aligned}$$

Give your answer accurate to four decimal places. State why you expect convergence using the Gauss-Seidel iteration for this system?

(14 marks)

 Question 2 continued on next page

(ii) Given the matrix

$$A = \begin{pmatrix} 2 & 4 & 3 \\ 1 & 4 & 1 \\ 2 & 1 & 1 \end{pmatrix},$$

perform three iterations with the power method to find an approximation of the largest eigenvalue starting from an initial guess $[1, 1, 1]^T$. Give your final estimate of the largest eigenvalue accurate to three decimal places. (5 marks)

(iii) For function $y(x)$, derive the first three non-zero terms of the Taylor series solution to the ordinary differential equation

$$y' = y^3 + x^2 - 4$$

subject to the initial condition $y(1) = 1$.

Hint: The Taylor series for a function $y(x)$ around a point $x = x_0$ is given by

$$y(x) = y(x_0) + \frac{(x - x_0)}{1!}y'(x_0) + \frac{(x - x_0)^2}{2!}y''(x_0) + \frac{(x - x_0)^3}{3!}y'''(x_0) + \ldots$$

(6 marks)

3 (i) The following data, $(0.1, 0.29)$, $(0.2, 0.33)$, $(0.3, 0.39)$, $(0.4, 0.45)$ and $(0.5, 0.53)$ can be fitted to the function $y = ae^{bx}$ where a and b are constants. By employing a suitable transformation, use the least squares linear fit to compute the values of a and b accurate to three decimal places.

Hint: A polynomial of degree n can be expressed by the following sum,

$$P_n(x) = \sum_{j=0}^{n} a_j x^j.$$

In the least squares sense, a polynomial of degree n can be fitted to data points $(x_i, f(x_i))$, where $i = 0, 1, 2, \ldots, m$ and $m > n$. Assuming that the x_i values are free of errors, the normal equations used in the process of a least squares fit for a polynomial of degree n are

$$\sum_{i=0}^{m} \left(\sum_{j=0}^{n} a_j x_i^{j+k} \right) = \sum_{i=0}^{m} x_i^k f_i, \quad k = 0, 1, 2, \ldots, n.$$

(15 marks)

 Question 3 continued on next page

3 (continued)

(ii) It is required to find the interpolating polynomial, $P_3(x)$, through the points $(0,3)$, $(1,2)$, $(2,7)$ and $(4,59)$. Implementing the Lagrange interpolating formula calculate $L_0(x)$, $L_1(x)$, $L_2(x)$, $L_3(x)$ and hence find the interpolated value of $P_3(2.9)$ accurate to three decimal places.

Hint: The Lagrange interpolation polynomial of least degree which passes through $(n+1)$ points (x_i, f_i), $i = 0, 1, 2, \ldots, n$ is

$$P_n(x) = \sum_{i=0}^{n} L_i(x) f_i$$

where

$$L_i(x) = \frac{(x - x_0)(x - x_1)\ldots(x - x_{i-1})(x - x_{i+1})\ldots(x - x_n)}{(x_i - x_0)(x_i - x_1)\ldots(x_i - x_{i-1})(x_i - x_{i+1})\ldots(x_i - x_n)}$$

and $f_i = f(x_i)$.

(10 marks)

4 (i) Sketch the feasibility region and find the maximum value of the function

$$f(x, y) = 7x + 6y$$

subject to the constraints

$$4x + 3y \leq 12, \quad 2x \leq 5, \quad y \leq 3, \quad x, y \geq 0.$$

(10 marks)

(ii) A factory makes two types of nails, both requiring specific quantities of materials A and B. Every 1000 nails of type 1 uses 1 kilogram of A and 3 kilograms of B, while every 1000 nails of type 2 uses 5 kilograms of A and 2 kilograms of B. Only 5 kilograms of A and 6 kilograms of B are available every day and the profit on the nails of type 2 is three times that on type 1.

Formulate the linear programming problem and use the graphical method to determine the number of nails of each type that should be produced to obtain the maximum daily profit. On the graph, clearly show the feasibility region and the line of constant revenue through the point of maximum daily profit. (15 marks)

End of Question Paper