The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2013-2014**

Vectors and Fluids **2 hours**

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

1 (i) A vector field is given by $\boldsymbol{F} = y^2 \sin x\, \boldsymbol{i} + 2y(2 - \cos x)\boldsymbol{j}$. Calculate

$$\boldsymbol{\nabla} \cdot \boldsymbol{F}, \quad \boldsymbol{\nabla} \times \boldsymbol{F},$$

and verify that the following result holds for this case:

$$(\boldsymbol{F} \cdot \boldsymbol{\nabla})\, \boldsymbol{F} = \frac{1}{2} \boldsymbol{\nabla}(\boldsymbol{F} \cdot \boldsymbol{F})\,.$$

(9 marks)

Deduce that there must exist a scalar $\phi = \phi(x, y, z)$ such that $\boldsymbol{F} = \boldsymbol{\nabla}\phi$. Determine the function ϕ that satisfies $\phi = 0$ at the point $(x, y) = (0, 1)$.

(7 marks)

(ii) Let $s = \sqrt{x^2 + y^2}$, $f = f(s)$ and $\boldsymbol{F} = \dfrac{f(s)}{s^2}(x\boldsymbol{i} + y\boldsymbol{j})$. Show that

$$\frac{\partial s}{\partial x} = \frac{x}{s},$$

and that $\boldsymbol{\nabla} \cdot \boldsymbol{F}$ can be expressed in terms of s and f', where $'$ denotes the derivative with respect to s.

(9 marks)

2 (i) Invoke the summation convention and let

$$\begin{aligned} a_{11} &= 3, & a_{12} &= 2, & a_{13} &= 5, \\ a_{21} &= 0, & a_{22} &= 4, & a_{23} &= 1, \\ a_{31} &= 1, & a_{32} &= 7, & a_{33} &= 6. \end{aligned}$$

Calculate, (a) a_{ii}, (b) $a_{3j}a_{j2}$, and (c) $\epsilon_{2jk}a_{jk}$.

(6 marks)

(ii) Find an expression for the i^{th}-component of $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A})$. *Using suffix notation* and the property

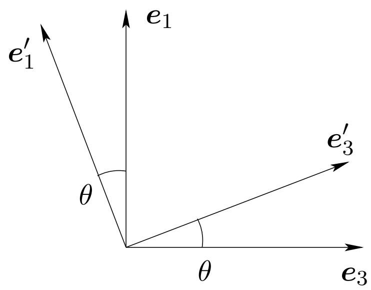
$$\epsilon_{ijk}\,\epsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}\,,$$

show that $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A})$ can be expressed in terms of $\boldsymbol{\nabla}^2\boldsymbol{A}$ and $\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A})$.

(11 marks)

(iii) Let L be a 3×3 matrix. State the conditions on L for it to be a valid matrix of transformation, representing a rotation of frames about a common origin.

A planar rotation takes an undashed frame to a dashed frame according to the following diagram:



By definition, the elements of L are given by $l_{ij} = \boldsymbol{e}'_i \cdot \boldsymbol{e}_j$. Identify the axis of rotation and construct the matrix L for this case.

(8 marks)

3 The spherical polar coordinates are related to the Cartesian system by

$$x = r\sin\theta\cos\phi\,, \quad y = r\sin\theta\sin\phi\,, \quad z = r\cos\theta\,.$$

(i) Find expressions for δx, δy and δz in terms of r, θ, ϕ, δr, $\delta\theta$ and $\delta\phi$.

Use your results to find an expression for

$$\delta l^2 = \delta x^2 + \delta y^2 + \delta z^2,$$

in terms of r, θ, ϕ, δr, $\delta\theta$ and $\delta\phi$. (You may assume that there are no 'cross terms'.) From this expression, determine h_1, h_2 and h_3 (standard notation) in terms of r and θ.

(10 marks)

(ii) The vector field $\boldsymbol{F}$ is given by

$$\boldsymbol{F} = -y^3\boldsymbol{i} + xy^2\boldsymbol{j}\,.$$

Sketch the surface S defined by $x^2+y^2+z^2 = a^2$ with $z \le 0$, and mark the bounding circuit C. Verify that Stokes' Theorem, namely

$$\int_S \boldsymbol{\nabla}\times\boldsymbol{F}\cdot\mathbf{d}\boldsymbol{S} = \oint_C \boldsymbol{F}\cdot\mathbf{d}\boldsymbol{l}\,,$$

holds for this case.

(15 marks)

4 A flow $\boldsymbol{u}$ inside a cylinder of radius a is given by the stream function

$$\psi = -\,U\left(r + \beta r^2\right)\cos\theta\,,$$

for a fluid of density ρ, where in cylindrical polar coordinates (r, θ, z),

$$\boldsymbol{u} = \boldsymbol{\nabla}\times(\psi\hat{\boldsymbol{z}}) = \frac{1}{r}\frac{\partial\psi}{\partial\theta}\,\hat{\boldsymbol{r}} - \frac{\partial\psi}{\partial r}\,\hat{\boldsymbol{\theta}},$$

(i) Show that the velocity field $\boldsymbol{u}$ satisfies the incompressibility condition, and find $\boldsymbol{u}$.

Determine the flow at the origin and use the appropriate boundary condition on $r = a$ to determine β.

(8 marks)

(ii) Find the stagnation points and sketch the flow.

(8 marks)

(iii) Given that the pressure at the origin is p_0, find an expression for the pressure p on the boundary surface.

The force over the whole boundary surface S is given by

$$-\int_S p\;\mathbf{d}\boldsymbol{S}\,.$$

By integrating its y-component, calculate the force in the y direction per unit length of the cylinder.

(9 marks)

5 (i) It is given that $\boldsymbol{\nabla} \cdot (\phi\, \boldsymbol{\nabla}\phi) = \phi\, \nabla^2\phi + \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\phi$. State Gauss' Theorem and show that if ϕ satisfies the Laplace equation, then

$$\int_V \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\phi \; \mathrm{d}V \;=\; \int_S \phi\, \boldsymbol{\nabla}\phi \cdot \mathbf{d}\boldsymbol{S}\,,$$

where $\mathbf{d}\boldsymbol{S}$ is a normal vector pointing out of the volume of interest, V.

(5 marks)

(ii) The flow of an incompressible fluid around an expanding sphere is described by the velocity potential

$$\phi = -\,\frac{R^2\,\dot{R}}{r}\,,$$

where $R = R(t)$ is the radius of the sphere, and far from the sphere the fluid is at rest.

In spherical polar coordinates we have $h_1 = 1$, $h_2 = r$ and $h_3 = r\sin\theta$. Calculate the velocity $\boldsymbol{u}$ and verify that the appropriate boundary conditions are satisfied.

(8 marks)

(iii) The kinetic energy of the fluid around the sphere in part (ii) is given by

$$KE = \frac{1}{2}\,\rho \int_V \boldsymbol{u} \cdot \boldsymbol{u}\, \mathrm{dV}\,.$$

By integration over the region outside the sphere, calculate the total kinetic energy of the flow.

Show that Gauss' theorem holds for this case, using the result in part (i).

(12 marks)

End of Question Paper