The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2013–2014**

Probability Modelling **2 hours**

*Candidates should attempt **ALL** five questions.*
The maximum marks for the various parts of the questions are indicated.
The paper will be marked out of 100. (Q1–19; Q2–21; Q3–29; Q4–14; Q5–17)

1 A non-delayed renewal process has u_n, the probability of a renewal at time n, satisfying

$$u_n = \frac{1}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^n$$

for $n \geq 1$.

(a) Find the generating function $U(s) = \sum_{n=0}^{\infty} u_n s^n$ for $0 \leq s < 1$. ***(5 marks)***

(b) Show that the generating function $F(s) = \sum_{n=1}^{\infty} f_n s^n$, where f_n is the probability that the first renewal occurs at time n, satisfies

$$F(s) = \frac{s^2}{2-s}$$

for $0 \leq s < 1$. [You may use results from the course.] ***(5 marks)***

(c) Explain why $f_1 = 0$, and find an expression for f_n for $n \geq 2$. ***(5 marks)***

(d) What is the expected time until the first renewal? ***(4 marks)***

2 A discrete time Markov chain (X_n) has state space $S = \{1,2,3,4,5,6\}$ and transition matrix given by

$$\begin{pmatrix} 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 1/3 & 0 & 0 & 2/3 & 0 & 0 \\ 1/20 & 0 & 0 & 0 & 0 & 19/20 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

(a) Find the communicating classes of this Markov chain. ***(4 marks)***

(b) For each state of the Markov chain, state whether it is persistent or transient, and give its period. ***(10 marks)***

(c) What is the three step transition probability $p_{33}^{(3)}$? Use your answer to prove by induction that $P(X_{3n} = 3) = (19/20)^n$ for all positive integers n if the chain starts in state 3. ***(7 marks)***

3 (a) Let (X_n) be a Markov chain on the state space $\{1,2,3\}$ with transition matrix

$$P_X = \begin{pmatrix} 1/4 & 1/4 & 1/2 \\ 1/2 & 0 & 1/2 \\ 2/3 & 1/3 & 0 \end{pmatrix}.$$

(i) Find the unique stationary distribution of the chain. ***(4 marks)***

(ii) Show that, for each $i \in \{1,2,3\}$, $P(X_n = i)$ converges to a limit as $n \to \infty$, regardless of the starting state of the chain, and give the value of the limit in each case. (You may use results from the course.) ***(10 marks)***

(b) Let (Y_n) be a Markov chain on the state space $\{1,2,3,4\}$ with transition matrix

$$P_Y = \begin{pmatrix} 1/4 & 1/4 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 2/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(i) Find all stationary distributions of this Markov chain. ***(5 marks)***

(ii) Assume that the chain starts in state 1. Using your answer to (a), or otherwise, describe the limiting behaviour of the probabilities $P(X_n = i)$ as $n \to \infty$ for $i = 1,2,3,4$. ***(7 marks)***

(iii) Describe the long run behaviour of the chain if it starts in state 4. ***(3 marks)***

4 A building has four floors, labelled 1,2,3 and 4. A lift in the building may be at one of the floors, or it may be out of service. Following the movement of the lift in discrete time, at each time step the lift, if it is currently at a floor, will go out of service with probability $1/4$, and otherwise will move to one of the other floors, each with equal probability. If the lift is currently out of service, at the next time step it will remain out of service with probability $1/2$ and otherwise will move to floor 1.

Label the possible states of the system as $\{0, 1, 2, 3, 4\}$ where 0 represents out of service and for $i = 1, 2, 3, 4$ i represents being at floor i, and model the behaviour of the lift as a discrete time Markov chain with state space $\{0, 1, 2, 3, 4\}$.

(a) Give the transition matrix of the Markov chain. ***(5 marks)***

(b) Find the expected number of steps until the lift reaches floor 4

(i) if it is at floor 1 at time 0;

(ii) if it is out of service at time 0. ***(9 marks)***

5 Team A and team B are playing a football match which lasts for 90 minutes. Goals scored by team A occur as a Poisson process with rate $\lambda_A = 1/60$ per minute, while goals scored by team B occur as a variable rate Poisson process with rate per minute given by $\lambda_B(t) = t/3000$, where t is the number of minutes since the start of the match. Assume that the two processes are independent.

(a) Find the probability that the two teams both score 2 goals in the match. ***(5 marks)***

(b) What is the distribution of the total number of goals scored in the game up to time t minutes, for $0 < t \leq 90$? ***(5 marks)***

(c) Given that team B score two goals in total in the match, what is the probability that they score exactly one goal in the first 45 minutes of the match? ***(7 marks)***

End of Question Paper