The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2013–2014**

Rings and Groups **2 hours**

*Answer **all four questions**.*
You should justify your answers carefully unless the question states otherwise.

1 (i) Write down the units in $\mathbb{Z}_{20}$, justifying your answer. Is the multiplicative group they form cyclic? ***(7 marks)***

(ii) Is it true that the units in the ring $\mathbb{Z}_{20}[x]$ are the same as those in $\mathbb{Z}_{20}$, expressed as constant polynomials? Prove this or give a counterexample. ***(4 marks)***

(iii) Let R be a ring. Prove that an element r of R cannot be both a unit and a zero-divisor. ***(4 marks)***

Additional marks for rigour and presentation. ***(5 marks)***

2 (i) Let d be a square-free integer with $d \neq 1$. Recall that the norm of an element $r = a + b\sqrt{d}$ of $\mathbb{Z}[\sqrt{d}]$, where $a, b \in \mathbb{Z}$, is given by

$$\mathcal{N}(a + b\sqrt{d}) = |a^2 - b^2 d|.$$

Show that $\mathbb{Z}[\sqrt{-5}]$ has no element of norm 7. Hence show that any element of norm 49 is irreducible. ***(6 marks)***

(ii) By considering the element $2 + 3\sqrt{-5}$ in $\mathbb{Z}[\sqrt{-5}]$, express 49 as a product of irreducible factors in two different ways, and deduce that $\mathbb{Z}[\sqrt{-5}]$ is not a unique factorisation domain. You may use that fact that an element of $\mathbb{Z}[\sqrt{-5}]$ is a unit if and only if its norm is 1. Justify your answer. ***(6 marks)***

(iii) Let R be a unique factorisation domain and let S be a subring of R. Is S necessarily a unique factorisation domain? Justify your answer briefly. ***(3 marks)***

Additional marks for rigour and presentation. ***(5 marks)***

3 Let G be a group of order 55 with trivial centre.

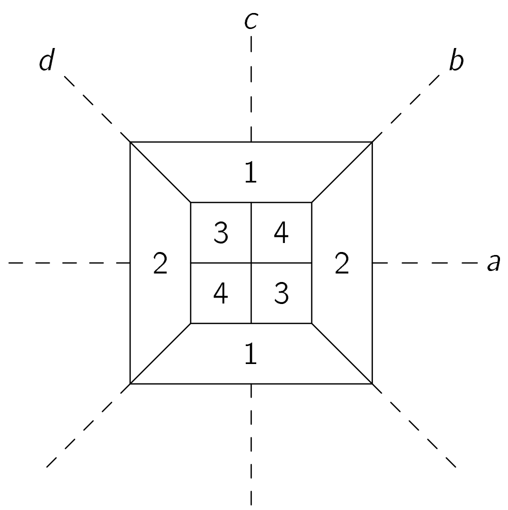
(i) Find the class equation of G, justifying your answer. ***(3 marks)***

(ii) What are the possible orders of elements of G? How many elements of each order are there? ***(7 marks)***

(iii) Let $h \in G$ be an element of order 11. Let $H = \langle h \rangle$, the subgroup generated by h. Use the class equation to show that H is a normal subgroup of G. Show that H is the only normal subgroup of G, other than the trivial group $\{e\}$ and G itself. ***(5 marks)***

Additional marks for rigour and presentation. ***(5 marks)***

4 Consider the square divided into regions labelled 1, 2, 3 and 4, and lines of symmetry labelled a, b, c and d, as below.



Recall that D_4 is the group of symmetries of the square, and write e for the identity of the group, r for rotation through $\frac{\pi}{2}$ anti-clockwise, and s_i for reflection in the line i, for $i = a, b, c, d$. Observe that D_4 acts on the numbered regions of the square, inducing a homomorphism $f : D_4 \longrightarrow S_4$.

(i) List the elements of D_4, and write down the effect of f on each of them. ***(6 marks)***

(ii) Write down the kernel and image of the homomorphism f. Is the image of f isomorphic to the cyclic group of order 4 or the Klein 4-group? Is f injective? Is f surjective? ***(4 marks)***

(iii) State, without proof, the First Isomorphism Theorem for groups. ***(2 marks)***

(iv) What does the First Isomorphism Theorem tell us in this case? ***(3 marks)***

Additional marks for rigour and presentation. ***(5 marks)***

End of Question Paper