The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — **Spring Semester 2013–2014**

Vector spaces and Fourier theory — **2 hours**

Attempt all of the questions. Each of the four questions is worth 25 marks, with the specific allocation shown in brackets.

1 Throughout this question, V will denote a vector space over a field $\mathbb{F}$.

(i) Let B be a set contained in V. Define what it means for B to

(a) be a spanning set of V;

(b) be linearly independent;

(c) be a basis for V. ***(8 marks)***

(ii) (a) Show that if U and W are subspaces of V then $U \cap W$ is a subspace of V.

(b) Let U, W be subspaces of $\mathbb{R}[x]_{\leq 3}$ such that $U + W = \mathbb{R}[x]_{\leq 3}$ with $\dim(U) = \dim(W) = 3$. Find $\dim(U \cap W)$. ***(8 marks)***

(iii) Recall that a function $f(x) \in C[-1, 1]$ is said to be *even* if $f(x) = f(-x)$ for all $x \in [-1, 1]$ and *odd* if $f(x) = -f(-x)$ for all $x \in [-1, 1]$. Define F to be the set of all functions that are either even or odd. That is,

$$F := \{f(x) \in C[-1, 1] \mid f(x) \text{ is even or odd}\}.$$

Is F a subspace of $C[-1, 1]$? You should justify your answer by providing a proof or a counterexample. ***(5 marks)***

(iv) Let B be a set contained in V. Define U to be the intersection of all subspaces containing B. Show that $U = \mathrm{sp}(B)$. ***(4 marks)***

2 (i) Let V and W will be vector spaces over a field $\mathbb{F}$. Let $L : V \to W$ be a linear map.

(a) Define what it means for L to be invertible.

(b) Define $\ker(L)$.

(c) Define $\operatorname{im}(L)$. ***(8 marks)***

(ii) Let $B, C \subseteq M_{2\times 2}(\mathbb{R})$ be the ordered bases for $M_{2\times 2}(\mathbb{R})$ defined by

$$B := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$
$$C := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}.$$

(a) Calculate the change of basis matrix $[I]_B^C$.

(b) Calculate the coordinate vector $[A]_C$ where A is the 2×2 matrix given by

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(8 marks)

(iii) State the rank-nullity theorem and show that a linear operator $L : V \to V$ on a finite dimensional space V is onto if and only if $\ker(L) = \{\mathbf{0}\}$. ***(5 marks)***

(iv) Let $L : \mathbb{R}[x]_{\leq n} \to \mathbb{R}[x]_{\leq n}$ be the linear operator defined by

$$p(x) \mapsto p(-x).$$

Find $\det(L)$ in terms of n. ***(4 marks)***

3 (i) Let V be a vector space over $\mathbb{R}$. Define what it means to say that $\langle , \rangle : V \times V \to \mathbb{R}$ is a real inner product on V. ***(8 marks)***

(ii) Let V be a real inner product space.

(a) For all $\mathbf{u}, \mathbf{v} \in V$ show that

$$\langle \mathbf{u}, \mathbf{v} \rangle = \tfrac{1}{4}(\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2).$$

(b) Conclude from part (a) that $\mathbf{u}, \mathbf{v}$ are orthogonal if and only if $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}-\mathbf{v}\|$. ***(8 marks)***

(iii) State the Cauchy-Schwarz inequality and show that

$$\int_0^1 x^{40} e^x \mathrm{dx} \leq \frac{1}{9}\sqrt{\frac{e^2-1}{2}}.$$

(5 marks)

(iv) Let L be a linear operator on an inner product space V with an adjoint L^*. Show that if $LL^* = L^*L$ then $\|L(\mathbf{v})\| = \|L^*(\mathbf{v})\|$ for all $\mathbf{v} \in V$. ***(4 marks)***

4 (i) Let V be a real inner product space. Define what it means to say

(a) that $\mathbf{u}, \mathbf{v} \in V$ are orthogonal;

(b) that $\mathbf{v} \in V$ is a unit vector;

(c) $B \subseteq V$ is an orthonormal basis. ***(8 marks)***

(ii) Consider the real inner product space $\mathbb{R}[x]_{\leq 2}$, with inner product defined by

$$\langle f(x), g(x) \rangle := \int_{-1}^1 f(x)g(x)\mathrm{dx}.$$

By applying the Gram-Schmidt orthogonalisation process to the basis $\{1, x, x^2\}$ find an orthogonal basis for $\mathbb{R}[x]_{\leq 2}$. ***(8 marks)***

(iii) Find the $a, b, c \in \mathbb{R}$ that realise

$$\min \left\{ \int_{-1}^1 (|x| + a + bx + cx^2)^2 \mathrm{dx} \; : \; a, b \in \mathbb{R} \right\}.$$

(5 marks)

(iv) State the convergence of Fourier series theorem for continuous 2π periodic functions. ***(4 marks)***

End of Question Paper