The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2013–2014**

Nonlinear Mathematics **2 hours**

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

Throughout the paper $\mathrm{I}(a,b)$ *denotes the open interval* $\{t \in \mathbb{R} \mid a < t < b\}$ *and* I *denotes an open interval in* $\mathbb{R}$ *with unspecified endpoints.*

1 (i) Define $\varphi\colon \mathbb{R}^4 \to \mathbb{R}^3$ by $\varphi(x,y,z,t) = (u,v,w)$, where

$$u = 2(xt - yz), \quad v = 2(xy + zt), \quad w = x^2 - y^2 + z^2 - t^2.$$

Also define $F\colon \mathbb{R}^3 \to \mathbb{R}$ by $F(u,v,w) = u^2 + v^2 + w^2$.

(a) Find the composite $F \circ \varphi\colon \mathbb{R}^4 \to \mathbb{R}$, simplifying your answer as much as possible.

(b) Let $S = \{(u,v,w) \in \mathbb{R}^3 \mid u^2 + v^2 + w^2 = 1\}$. Find $\varphi^{-1}(S)$.

(8 marks)

(ii) Let $(a,b) \in \mathbb{R}^2$ and let $r > 0$. Define the open ball $B((a,b),r)$.

Define what it means for a set $M \subseteq \mathbb{R}^2$ to be open. **(5 marks)**

Let $(a,b) \in \mathbb{R}^2$ and define $M = \mathbb{R}^2 \backslash \{(a,b)\}$. Prove, directly from your definition, that M is an open set. **(5 marks)**

(iii) Define $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $\varphi(x,y) = (y, xy)$. Find the image of φ and show that it is not an open set. **(7 marks)**

2 (i) (a) Define what it means for a function $F\colon M \to \mathbb{R}$, where $M \subseteq \mathbb{R}^2$ is an open set, to be continuous at $(a,b) \in M$.

(b) Define what it means for a function $F\colon M \to \mathbb{R}$, where $M \subseteq \mathbb{R}^2$ is an open set, to be C^1. **(7 marks)**

(ii) (a) Let $H\colon M \to \mathbb{R}$ be a continuous function defined on an open set $M \subseteq \mathbb{R}^2$ and let $(a,b) \in M$.

If $H(a,b) \neq 0$ show that there is an $r > 0$ such that $B((a,b),r) \subseteq M$ and $H(x,y) \neq 0$ for all $(x,y) \in B((a,b),r)$. **(5 marks)**

(b) Let $F\colon M \to \mathbb{R}$ and $G\colon M \to \mathbb{R}$ be C^1 functions defined on an open set $M \subseteq \mathbb{R}^2$. Let $(a,b) \in M$ be such that

$$\begin{vmatrix} \dfrac{\partial F}{\partial x}(a,b) & \dfrac{\partial F}{\partial y}(a,b) \\ \dfrac{\partial G}{\partial x}(a,b) & \dfrac{\partial G}{\partial y}(a,b) \end{vmatrix} \neq 0.$$

Show that there is $r > 0$ such that $B((a,b),r) \subseteq M$ and

$$\begin{vmatrix} \dfrac{\partial F}{\partial x}(x,y) & \dfrac{\partial F}{\partial y}(x,y) \\ \dfrac{\partial G}{\partial x}(x,y) & \dfrac{\partial G}{\partial y}(x,y) \end{vmatrix} \neq 0$$

for all $(x,y) \in B((a,b),r)$. **(7 marks)**

(iii) Define $F\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$F(x,y) = \begin{cases} \dfrac{x^2y}{x^4+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0). \end{cases}$$

Show that F is not continuous at $(0,0)$, stating clearly any theorem from the course that you use. **(6 marks)**

3 (i) State the triangle identity for elements of $\mathbb{R}^2$. **(3 marks)**

(ii) Let $F\colon M \to \mathbb{R}$ and $G\colon M \to \mathbb{R}$ be functions defined on an open set $M \subseteq \mathbb{R}^2$ which are continuous at a point $(a,b) \in M$.

Prove that FG is continuous at (a,b). **(14 marks)**

(iii) Let m, n, p and q be non-zero integers, and let $\lambda \in \mathbb{R}$, $\lambda \neq 0$.

Write $M = \{(x,y) \mid x > 0, y > 0\}$ and define $\varphi\colon M \to \mathbb{R}^2$ by

$$\varphi(x,y) = (\lambda x^m y^n, \lambda x^p y^q).$$

For which values of m, n, p, q and λ is the determinant of the derivative of φ equal to 1 for all $(x,y) \in M$? **(8 marks)**

4 (i) Let $u, v \in \mathbb{R}$ and consider the cubic equation

$$t^3 + ut + v = 0. \qquad (*)$$

Suppose that $(*)$ has three real roots x, y, z, some or all of which may be equal.

(a) Prove that

$$u = -x^2 - xy - y^2, \qquad v = x^2y + xy^2.$$

(3 marks)

(b) Define $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\varphi(x, y) = (-x^2 - xy - y^2, x^2y + xy^2).$$

Determine the set E of points (x, y) at which $\det D(\varphi)(x, y) = 0$. **(4 marks)**

(c) Determine the image $\varphi(E)$ of E under φ, expressing your answer in terms of u and v. **(4 marks)**

(ii) Let $I \subseteq \mathbb{R}$ be an open interval and let $M \subseteq \mathbb{R}^3$ be an open set. Let $\alpha\colon I \to \mathbb{R}^3$ and $\varphi\colon M \to \mathbb{R}^2$ be C^1 maps such that $\alpha(t) \in M$ for all $t \in I$.

(a) Write down (but do not prove) the Chain Rule for $\varphi \circ \alpha$.

(b) Now assume that $D(\varphi \circ \alpha)(t) = (0, 0)$ for all $t \in I$ and that there is a point $t_0 \in I$ such that

$$\det \begin{bmatrix} \dfrac{\partial \varphi_1}{\partial x}(x_0, y_0, z_0) & \dfrac{\partial \varphi_1}{\partial y}(x_0, y_0, z_0) \\ \dfrac{\partial \varphi_2}{\partial x}(x_0, y_0, z_0) & \dfrac{\partial \varphi_2}{\partial y}(x_0, y_0, z_0) \end{bmatrix} \neq 0$$

where $(x_0, y_0, z_0) = \alpha(t_0)$ and $\varphi = (\varphi_1, \varphi_2)$.

Express $\alpha_1'(t_0)$ and $\alpha_2'(t_0)$ in terms of partial derivatives of φ and $\alpha_3'(t_0)$. **(14 marks)**

5 (i) State the Chain Rule as it applies to a C^1 map $\psi\colon M \to N$ and a C^1 function $H\colon N \to \mathbb{R}$, where M and N are open subsets of $\mathbb{R}^2$. **(3 marks)**

(ii) Let M and N be open subsets of $\mathbb{R}^3$. Define what it means for a map $\varphi\colon M \to N$ to be a diffeomorphism. **(3 marks)**

(iii) State carefully and in full the Local Diffeomorphism Theorem for maps $M \to N$ where M and N are open sets in $\mathbb{R}^3$. **(6 marks)**

(iv) Let $S\colon \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function, written as $S(x,u)$, for which the two partial derivatives

$$H(x,u) = \frac{\partial S}{\partial x}(x,u), \quad \text{and} \quad K(x,u) = \frac{\partial S}{\partial u}(x,u),$$

are also C^1. You may use without proof the fact that $\dfrac{\partial H}{\partial u} = \dfrac{\partial K}{\partial x}$ everywhere.

Now suppose there is a point $(x_0, u_0) \in \mathbb{R}^2$ at which $\dfrac{\partial^2 S}{\partial u \partial x}(x_0, u_0) \neq 0$.

Define $\widetilde{H}\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\widetilde{H}(x,u) = (x, H(x,u)).$$

Show that the Local Diffeomorphism Theorem can be applied to $\widetilde{H}$ at (x_0, u_0), stating your conclusion in full.

Deduce that there is an open set $N_1 \subseteq \mathbb{R}^2$ and a C^1 function $F\colon N_1 \to \mathbb{R}$, written $u = F(x,y)$, such that

$$H(x, F(x,y)) = y$$

for all $(x,y) \in N_1$. **(8 marks)**

(v) Using (i), or otherwise, show that

$$\frac{\partial H}{\partial x} + \frac{\partial H}{\partial u}\frac{\partial F}{\partial x} = 0, \qquad \frac{\partial H}{\partial u}\frac{\partial F}{\partial y} = 1.$$

(5 marks)

End of Question Paper