The University Of Sheffield.

**SCHOOL OF MATHEMATICS AND STATISTICS** **Spring Semester 2013–2014**

**Fluid Mechanics I** **2 hours**

*Answer all four questions.*

**1** Consider a flow between parallel plates $y = 0, h$. The upper plate $y = h(> 0)$ is moving with a constant speed $V(> 0)$ in the $x$-direction and the lower plate is at rest. We look for a steady flow of the form

$$\mathbf{v} = (u(x, y), 0, 0).$$

This is a two-dimensional problem and no quantities depend on $z$. We assume that the pressure does not depend on $x$.

(i) By using the continuity equation, show that $u$ is actually a function of $y$ only. ***(2 marks)***

(ii) Use the Navier-Stokes equations to show that

$$\frac{d^2u}{dy^2} = 0 \text{ and } \frac{dp}{dy} = 0.$$

***(8 marks)***

(iii) Determine $u$ by taking into account the boundary conditions.

***(4 marks)***

(iv) Compute the wall shear stress at $y = 0$ and confirm its physical dimension is consistent. ***(5 marks)***

(v) Compute the vorticity of the flow and state in which direction it points. ***(3 marks)***

(vi) Compute the volume flux (that is, the net volume passing between the plates per unit time per unit width) of the flow. ***(3 marks)***

**2** (i) Suppose that an incompressible velocity field in an inviscid fluid is given by

$$\mathbf{v} = (\alpha x - \Omega y, -y + \Omega x, 0)$$

relative to Cartesian coordinates $(x, y, z)$, where $\alpha$ and $\Omega$ may be functions of time. Find $\alpha$ and the vorticity, and show that $\Omega$ must be constant with respect to time. ***(7 marks)***

(ii) The material derivative of vector $\mathbf{A}$ is defined as

$$\frac{D\mathbf{A}}{Dt} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{A},$$

where $\mathbf{u}$ denotes fluid velocity which is in general compressible.

If $\theta(x, t)$ is a scalar function of position and time show that in an inviscid fluid

$$\frac{D}{Dt}(\omega \cdot \nabla\theta) = (\omega \cdot \nabla)\frac{D\theta}{Dt}.$$

Hence deduce that if $\theta(x, t)$ is any scalar quantity which is conserved by all fluid elements moving with the flow, the $(\omega \cdot \nabla)\theta$ is also a constant.

***(8 marks)***

(iii) Using the result in (iii), prove the following acceleration formula

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + \nabla\left(\frac{|\mathbf{u}|^2}{2}\right) + (\nabla \times \mathbf{u}) \times \mathbf{u}.$$

***(10 marks)***

**3** Consider the two-dimensional Navier-Stokes equations and the continuity equation for an incompressible velocity field $\mathbf{u} = (u, v)$,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right),$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right),$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

where $\nu$ denotes kinematic viscosity, $p$ the pressure and $\rho$ uniform density.

(i) Show that the vorticity $\omega = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$ satisfies

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \nu\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right).$$

***(9 marks)***

(ii) Using a stream function $\psi$ such that

$$\mathbf{u} = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right)$$

show that the above equation can be written as

$$\frac{\partial \omega}{\partial t} + J(\omega, \psi) = \nu\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right),$$

where

$$J(\omega, \psi) = \frac{\partial \omega}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial \omega}{\partial y}\frac{\partial \psi}{\partial x}$$

denotes a Jacobian determinant. ***(3 marks)***

(iii) Show that for $\omega = F(\psi)$ with an arbitrary function $F$, the advection term vanishes.

***(3 marks)***

(iv) Consider

$$\psi = A(t)\cos\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi y}{l}\right),$$

where $l(> 0)$ is a constant length scale. Show that we can choose a function $F(\psi)$ as defined in (iii). Determine $A(t)$ in such a way that it is a solution of the two-dimensional Navier-Stokes equations. ***(10 marks)***

**4** (i) A fluid moves in a steady two-dimensional flow in the region defined by $x \geq 0,\ y \geq 0$. The boundary with equation $y = 0$ is occupied by a stationary flat plate. Given that the $x$-component of velocity $u \to U$ as $y \to \infty$, where $U$ is a constant, write down the expressions for
(a) the displacement thickness $\delta_1$ of the boundary layer,
and
(b) the momentum thickness $\delta_2$ of the boundary layer.

***(4 marks)***

(c) When the flow is given approximately by

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^m \text{ where } 0 < m < 1,$$

compute $\delta_1$ and $\delta_2$ explicitly and hence show which is the largest.

***(6 marks)***

(ii) Consider a steady two-dimensional flow past a semi-infinite solid boundary along $y = 0$ in the region $y \geq 0,\ x \geq 0$. Blasius's boundary layer equations for it are given by

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2},$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

Derive the energy equation for the boundary layer

$$\frac{d}{dx}\int_0^\infty \frac{u}{U}\left(1 - \frac{u^2}{U^2}\right) dy = \frac{2D}{\rho U^3},$$

where $u,\ v$ are the $x$- and $y$- components of the velocity and

$$D = \mu\int_0^\infty \left(\frac{\partial u}{\partial y}\right)^2 dy$$

is the dissipation rate of energy. The constant $U$ is the $x$-component of the velocity as $y \to \infty$. ***(15 marks)***

## End of Question Paper