The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2013-2014**

Milestones in Applied Mathematics II **2 Hours**

Answer all four questions.

1 For the following questions, you are given that the value of Planck's constant is $h = 6.626 \cdot 10^{-34}$Js and $\hbar = h/2\pi = 1.0546 \cdot 10^{-34}$Js; The speed of light is $c = 2.998 \cdot 10^{8}$m/s;

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right); \qquad \sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right).$$

(i) BBC Radio 4 broadcasts on a frequency of 200 kHz. Estimate the number of photons striking a 1 square metre aerial each second at a distance of 1000 km from a 200 kW transmitter.

How would your answer change if the aerial were on a space probe at a distance of 3000 million km from the earth? ***(13 marks)***

(ii) Consider the free-particle wave-functions

$$\psi_1(x,t) = A\left(e^{i(kx-\omega t)} + e^{-i(kx+\omega t)}\right);$$
$$\psi_2(x,t) = B\left(e^{i(kx-\omega t)} - e^{-i(kx+\omega t)}\right);$$

where A and B are complex constants. For each wave-function, calculate the corresponding probability density and probability current.

Interpret your answers for probability current of ψ_2 physically in terms of streams of particles. ***(12 marks)***

2 A quantum mechanical particle of mass m moves in the potential

$$V(x) = \begin{cases} 0 & 0 < x < 2a; \\ \infty & \text{otherwise.} \end{cases}$$

(i) Find the normalized stationary state wavefunctions. ***(13 marks)***

(ii) For each stationary state, calculate the probability that the particle is in the interval $(0, a/2)$. ***(6 marks)***

(iii) For each stationary state, calculate the expectation value of the position of the particle. ***(6 marks)***

3 The Hamiltonian for a free particle is

$$H = \frac{P^2}{2m}.$$

At time $t = 0$, the free particle is in a normalized state ψ in which the expectation values of both position and momentum are zero. In this question you may use the result that, for any operator A,

$$i\hbar \frac{d}{dt} E_\psi(A) = \langle \psi | [A, H] \psi \rangle.$$

(i) Show that

$$\begin{aligned} [X, H] &= \frac{i\hbar}{m} P; \\ [X^2, H] &= \frac{i\hbar}{m} (XP + PX). \end{aligned}$$

(5 marks)

(ii) Show that $E_\psi(P)$ is zero for all time t, and that $E_\psi(P^2)$ is a constant.

(5 marks)

(iii) Show that

$$[XP + PX, H] = \frac{2i\hbar}{m} P^2$$

and hence that

$$\frac{d}{dt} E_\psi(XP + PX) = \frac{2}{m} E_\psi(P^2).$$

(5 marks)

(iv) By considering an operator A defined by

$$A = [X - E_\psi(X) I]^2 = X^2 - 2E_\psi(X) X + [E_\psi(X)]^2 I$$

where I is the identity operator, and using the results obtained in (i)–(iii), show that

$$\frac{d}{dt} [\Delta_\psi(X)]^2 = \frac{1}{m} E_\psi (XP + PX)$$

and hence that

$$[\Delta_\psi(X)]^2 = at^2 + bt + c$$

where a, b and c are constants. ***(10 marks)***

4 The Hamiltonian of a certain quantum system is given by

$$H = \frac{3\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and the states of the system are described by 2-dimensional complex vectors. At time $t = 0$, the state of the system is

$$\psi(0) = \alpha \begin{pmatrix} 1 \\ i \end{pmatrix}$$

where α is a positive, real constant.

(i) Find α so that this initial state is normalized. ***(4 marks)***

(ii) Write down the vector $\psi(t)$ giving the state of the system at time t. ***(4 marks)***

(iii) The system is acted on by operators A, B and C, which are given by:

$$A = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad B = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad C = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Show that $\psi(t)$ is an eigenvector of B and find the corresponding eigenvalue. Find $A\psi$ and $C\psi$. ***(5 marks)***

(iv) Calculate the expectation values of A, B and C in the state $\psi(t)$. ***(6 marks)***

(v) Find the dispersion of A in the state $\psi(t)$. ***(6 marks)***

End of Question Paper