Data provided: Formulae sheet

MAS340

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS Spring Semester 2013-2014

Optimization and Numerical Methods for Engineers **Two hours**

Marks will be awarded for your best FOUR answers

1 Let $f(x, y) = x^2 + x^2y + y^2$

(i) Starting from the initial point $(1, 1)$ use one iteration of the method of steepest descent to find the points (x_1, y_1) and determine the best choice of (x_1, y_1). ***(12 marks)***

(ii) Use one iteration of Newton's method, starting from (1, 1), to search for a stationary point of f (x, y). ***(7 marks)***

(iii) By solving directly for the critical points, find all turning points and classify them appropriately. ***(6 marks)***

2 Consider the following differential equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

and answer the following questions.

(i) Classify the differential equation as either elliptic, parabolic or hyperbolic. ***(1 mark)***

(ii) By selecting the correct finite difference schemes, and correct changes in notation, derive the following scheme for solving Equation (1).

$$u_{i,j+1} = r u_{i-1,j} + (1 - 2r) u_{i,j} + r u_{i+1,j}$$

(3 marks)

(iii) Using the finite difference scheme, as found in part (ii), solve the differential equation between $0 \leq x \leq 5, 0 \leq t \leq 1$ for

$$h = 1, \qquad k = \frac{1}{4}, \qquad c^2 = 1,$$

where h is the step size in x and k is the step size in time. The following boundary conditions

$$u(t, 0) = 25, \qquad u(t, 5) = 0, \qquad u(0, x) = 25 - x^2,$$

are also applied. ***(10 marks)***

(iv) Repeat the calculation with the following changes.

$$h = 1, \qquad k = \frac{1}{2}, \qquad c^2 = 4, \qquad 0 \leq t \leq 1.5.$$

(5 marks)

(v) Discuss your results from section (iv) in terms of their physicality and suggest improvements to the numerical scheme used. ***(6 marks)***

3 (i) Determine the L and U matrices for the following system

$$A\mathrm{x} = b, \qquad A = \begin{bmatrix} 5 & 3 & 4 \\ 3 & 4 & 7 \\ 4 & 3 & 9 \end{bmatrix}, \qquad b = \begin{bmatrix} 4 \\ 5 \\ 8 \end{bmatrix}.$$

(6 marks)

(ii) Determine L^{-1} and U^{-1}. ***(4 marks)***

(iii) Hence find the column vector x. ***(4 marks)***

(iv) Find the LU decomposition for the following tri-diagonal matrix

$$M = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}.$$

(6 marks)

(v) Using your results from part (iv) show M^{-1} to be

$$M^{-1} = \frac{1}{144} \begin{bmatrix} 55 & -21 & 8 & -3 & 1 \\ -21 & 63 & -24 & 9 & -3 \\ 8 & -24 & 64 & -24 & 8 \\ -3 & 9 & -24 & 63 & -21 \\ 1 & -3 & 8 & -21 & 55 \end{bmatrix}.$$

(5 marks)

4 A ship is to be loaded with a selection of goods of 3 types. The total weight of the load must not exceed 25 tonnes. The goods have weights and values as shown in the following table:

Type	Weight	Price
1	7	23
2	5	17
3	2	7

Using the dynamic programming algorithm, construct an appropriate table to find the combination of goods which gives the highest value for the load on the ship.

(25 marks)

5 (i) Write down the 'natural conditions' for a a cubic spline ***(2 marks)***

(ii) Briefly discuss the main differences between the 'natural conditions' and the 'fixed conditions' for cubic splines and the influence of the chosen conditions on the spline. ***(4 marks)***

(iii) Determine the cubic splines between the following data points

x	0	$\pi/3$	$2\pi/3$	π
f(x)	0	$\sqrt{3}/2$	$\sqrt{3}/2$	0

(17 marks)

(iv) Determine the value of the cubic spline at $x = 5\pi/6$. ***(2 marks)***

End of Question Paper

Formulae Sheet

Notation:

$$U(x_i,\ t_j) \equiv U_{i\,j}$$

Forward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t} \approx \frac{U_{i,j+1} - U_{ij}}{\Delta t}$$

Backward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t} \approx \frac{U_{ij} - U_{i,j-1}}{\Delta t}$$

Central difference formula for $\partial U/\partial x$:

$$\frac{\partial U}{\partial x} \approx \frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x}$$

Central difference formula for $\partial^2 U/\partial x^2$:

$$\frac{\partial^2 U}{\partial x^2} \approx \frac{U_{i+1,j} - 2U_{ij} + U_{i-1,j}}{\Delta x^2}$$