The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2013-2014**

Applicable Analysis **2 hours 30 minutes**

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

You may use the following results when answering questions on this paper.

Table of Laplace Transforms	
Function	*Laplace Transform*
$t^\alpha e^{bt} (\alpha > -1)$	$\dfrac{\Gamma(\alpha+1)}{(s-b)^{\alpha+1}}$
$\sin at$	$\dfrac{a}{s^2+a^2}$
$\cos at$	$\dfrac{s}{s^2+a^2}$
$f(t)e^{bt}$	$F(s-b)$
$f^{(n)}(t)$	$s^n F(s) - \sum_{k=1}^{n} f^{(k-1)}(0) s^{n-k}$
$tf(t)$	$-F'(s)$

1 (i) Define what is meant by the statement that $\int_a^{\infty} f(x)\,dx$ exists.

(2 marks)

Prove, **from your definition,** each of the following statements:

$$\text{(a)} \quad \int_0^{\infty} \frac{2}{x^2+2x+2}\,dx \quad \text{exists;}$$

$$\text{(b)} \quad \int_0^{\infty} \cos x\,dx \quad \text{does not exist.}$$

(5 marks)

(ii) State, without proof, the Comparison Test for convergence and divergence of integrals of the form $\int_a^{\infty} f(x)\,dx$. Your statement should include conditions under which the results are valid.

(4 marks)

Prove each of the following, stating any standard results you need to use:

$$\text{(a)} \quad \int_0^{\infty} \frac{2+\sin x}{x^2+2x+2}\,dx \quad \text{converges;}$$

$$\text{(b)} \quad \int_0^{\infty} \frac{1}{(1+x^7)^{1/7}}\,dx \quad \text{diverges.}$$

(7 marks)

(iii) Decide whether each of the following integrals converges or diverges and prove your assertions.

$$\text{(a)} \quad \int_0^{1} \frac{e^{-x}}{\sqrt{1-x}}\,dx\,;$$

$$\text{(b)} \quad \int_1^{\infty} \frac{1}{x\,\sqrt{(\ln x)}}\,dx\,.$$

(7 marks)

2 (i) State, without proof, the theorem concerning change of order in a repeated integral of the form

$$\int_c^d dy \int_a^\infty f(x,y)\,dx\,.$$

Your statement should include conditions under which the result holds. ***(2 marks)***

Show that for $\alpha > 0$,

$$\int_0^\infty e^{-\alpha x}\sin x\,dx \;=\; \frac{1}{\alpha^2+1}\,.$$ ***(4 marks)***

Let $0 \;<\; c \;<\; d$. Prove that

$$\int_c^d dy \int_0^\infty e^{-xy}\sin x\,dx \;=\; \int_0^\infty dx \int_c^d e^{-xy}\sin x\,dy\,.$$

Deduce that

$$\int_0^\infty \frac{(e^{-cx}-e^{-dx})\sin x}{x}\,dx \;=\; \tan^{-1} d - \tan^{-1} c\,.$$ ***(8 marks)***

(ii) Define the Γ function. ***(2 marks)***

Prove that

(a) $\displaystyle\int_0^\infty x^{-1/4}\,e^{-\sqrt{x}}\,dx \;=\; \sqrt{\pi}$ ***(3 marks)***

(b) $\displaystyle\int_1^\infty \left(\frac{\ln x}{x^2}\right)^2 dx \;=\; \frac{2}{27}\,.$ ***(6 marks)***

3 Define the Beta function. State, without proof, the relation between the Beta and Gamma functions. ***(3 marks)***

Prove that

$$B(x,y) = 2\int_0^{\pi/2} \cos^{2x-1}\theta \sin^{2y-1}\theta \, d\theta \qquad (x>0,\ y>0)$$

and

$$B(x,y) = \int_0^{\infty} \frac{u^{x-1}}{(1+u)^{x+y}} \, du \qquad (x>0,\ y>0).$$

(4 marks)

Prove each of the following, stating any standard results you need to use:

(a) $$\int_0^{\pi/2} \frac{\sqrt{\tan\theta}}{1+\tan^2\theta} \, d\theta = \frac{\pi\sqrt{2}}{8} \, ;$$

(b) $$\int_0^{\infty} \frac{x\sqrt{x}}{1+x^6} \, dx = \frac{\pi}{6} \operatorname{cosec}\left(\frac{5\pi}{12}\right) ;$$

(c) $$\int_0^{a} x^4 \left(a^3 - x^3\right)^{1/3} dx = \frac{2a^6\pi}{27\sqrt{3}} \quad (a>0).$$ ***(18 marks)***

4 (i) Define what is meant by the statement that $\int_0^\infty f(t)e^{-st}\,dt$ has abscissa of convergence c. ***(2 marks)***

Find the abscissa of convergence of $\displaystyle\int_0^\infty \frac{e^{-st}}{1+t}\,dt$, giving reasons for your answer. ***(5 marks)***

(ii) In each of the following cases, find the function continuous on $[0,\infty)$, with the given Laplace transform:

(a) $\dfrac{s+3}{s^2-1} \quad (s>1)\,;$

(b) $\dfrac{2}{s^2-2s+5} \quad (s>1)\,;$

(c) $\dfrac{2s^2-s+5}{(s+1)(s^2-2s+5)} \quad (s>1)\,.$

(8 marks)

(iii) Suppose the functions f and g are continuous on $[0,\infty)$. Define the convolution $f*g$.

State, without proof, a relation between $L(f*g)$, $L(f)$ and $L(g)$. ***(3 marks)***

Using Laplace transforms, find the function f continuous on $[0,\infty)$ such that

$$\int_0^t f(u)\cos(t-u)\,du \quad = \quad t-1+e^{-t} \quad (t>0).$$ ***(7 marks)***

5 (i) Using Beta functions, or otherwise, show that

$$\int_0^\infty \frac{1}{\sqrt{x}\,(\,s^2 + x^4\,)}\,dx \quad = \quad \frac{\pi}{4s^{7/4}\,\sin{(\pi/8)}}\,.$$ ***(6 marks)***

By considering

$$\int_0^\infty \frac{\cos{(x^2\,t)}}{\sqrt{x}}\,dx\,,$$

show that

$$\int_0^\infty \frac{\cos x^2}{\sqrt{x}}\,dx \quad = \quad \frac{\pi}{4\sin{(\frac{\pi}{8})}\,\Gamma(\frac{3}{4})}\,.$$ ***(7 marks)***

(ii) Find the solution of the differential equation

$$t\frac{d^2y}{dt^2} + 2(t+1)\frac{dy}{dt} + 2y = (1-2t)e^{-2t}$$

subject to the conditions $y(0) = 0,\; y'(0) = \frac{1}{2}$. ***(12 marks)***

End of Question Paper