The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2013–2014**

Knots and Surfaces **2 hours and 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.
Strings, pipe cleaners, shoe laces or similar aids for making knots may be used.

1 (i) Draw the three *Reidemeister moves*, explain what they represent and state *Reidemeister's Theorem.*

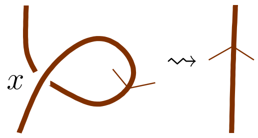
(8 marks)

(ii) A Gauss word is a string of numbers in which each number occurs twice and each number occurs once with a bar over it, e.g. $\overline{1}2\overline{2}1$. If k_p is an oriented knot diagram with a specified base point p on it then a Gauss word $g(k_p)$ is obtained in the following way. Label each of the crossings with a number. Starting at the base point, trace round the knot in the direction of the orientation and write down the label of each crossing as it is passed; if you pass the crossing on the under strand then put a bar over the label.

Give a Gauss word for the standard diagram of the positive trefoil. Show that this Gauss word can also be obtained from the standard diagram of the negative trefoil.

(4 marks)

Performing Reidemeister moves on a knot diagram gives rise to word operations on the associated Gauss words. For instance, performing the Reidemeister I move at a crossing labelled by x as follows (where the base point is elsewhere on the knot)



gives the word operation

$$Ax\overline{x}B \rightsquigarrow AB,$$

where A and B are strings of numbers representing the crossings not pictured.

Write down which word operations can arise from performing a Reidemeister II move, assuming that the specified base point is not in the disc region affected by the move. ***(5 marks)***

(iii) Recall that $\epsilon(x)$ denotes the sign of the crossing x. The Casson invariant of an oriented knot is defined as the following sum,

$$C(k) := \sum_{A\overline{x}ByCxD\overline{y}} \epsilon(x)\epsilon(y),$$

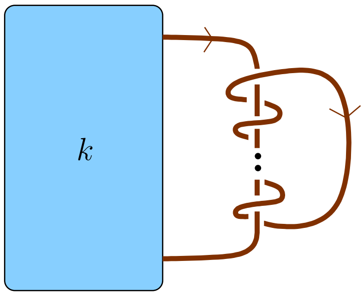
where you pick a base point p on the diagram and take the sum over all the crossing labels x and y which appear in the associated Gauss word $g(k_p)$ in the order $A\overline{x}ByCxD\overline{y}$ — here A, B, C and D are possibly empty strings of numbers.

Calculate the Casson invariant of the positive trefoil and the figure eight knot.

Prove that the Casson invariant is invariant under the Reidemeister II move, you may assume that the specified base point is not in the disc region affected by the move.

(8 marks)

2 Suppose that k is a link diagram with a chosen piece of string from one of its components. We may twist this together with an unknot, adding $2n$ negative crossings, for $n \geq 0$, to form $k^{(n)}$ as pictured.

$k^{(n)} :=$ 

(i) Draw $k^{(n)}$ when k is the unknot and $n = 0, 1, 2$. Identify the link when $n = 0$ and $n = 1$, thus write down the Jones Polynomial in these two cases. ***(7 marks)***

(ii) Show that for $n \geq 1$, the Jones Polynomial satisfies the equation

$$f[k^{(n)}] = A^8 f[k^{(n-1)}] + A^2(A^4 - 1)f[k].$$

(7 marks)

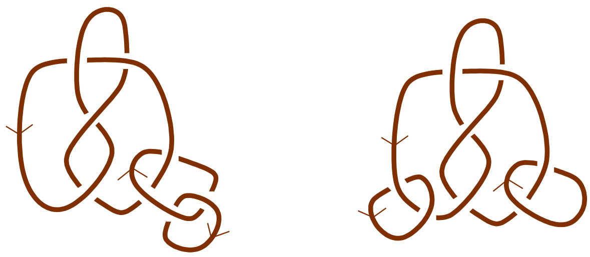
(iii) Write down a formula for $f[k^{(1)}]$ in terms of $f[k]$. ***(1 mark)***

(iv) Show by induction on n that, for $n \geq 1$,

$$f[k^{(n)}] = -A^2 \left(A^{8n} + \frac{A^{8n-4} + 1}{A^4 + 1} \right) f[k],$$

(4 marks)

(v) Write down the Jones Polynomials of the two links below in terms of the Jones Polynomial of the figure eight knot. Are the two links equivalent?



(6 marks)

3 State whether the following are true or false. Carefully justify your answer with a proof or counterexample as appropriate. Results from lectures may be used if correctly stated.

(i) Two orientable surface words of the same length represent homeomorphic surfaces.

(ii) If a surface word has a pair of letters with the same exponent then the associated surface is non-orientable.

(iii) If a surface M is obtained by identifying the two end circles of a cylinder then M is homeomorphic to a torus.

(iv) The open disk $\{(x, y) \mid x^2 + y^2 < 1\} \subset \mathbb{R}^2$ is a surface.

(v) The surface associated to the surface word *abcabc* is the connected sum of two non-orientable surfaces. ***(25 marks)***

4 (i) State the *inclusion/exclusion principle* for the Euler characteristic, and state the Euler characteristic of a *point*, a *line segment* and a *circle*. Using these together with the homeomorphism invariance of the Euler characteristic, find the Euler characteristic of a *disc*. ***(8 marks)***

If X is surface and X^{o} denotes X with an open disc removed then use the inclusion/exclusion principle to show that $\chi(X^{\text{o}}) = \chi(X) - 1$. If Y denotes another surface, then from this, obtain an expression for $\chi(X \# Y)$ in terms of $\chi(X)$ and $\chi(Y)$. ***(6 marks)***

(ii) State the formula for the Euler characteristic of a surface in terms of a covering pattern of the surface, explaining any symbols you use. Draw the tetrahedral triangulation of the sphere S^2 and use it to calculate the Euler characteristic of the sphere. ***(6 marks)***

A surface Z has a continuous map $Z \to S^2$ which is onto and which is 2-to-1 at all except four points where it is 1-to-1. Identify the surface Z. [Hint: you might think about the preimage of a triangulation of S^2.] ***(5 marks)***

End of Question Paper