The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2013–2014**

MAS346 Groups and Symmetry **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) Let X be a set. Prove that the set $S(X)$ of all bijections $f : X \to X$ is a group under composition of functions. ***(6 marks)***

(ii) Let G be a group. For any $a \in G$ define the map $l_a : G \to G$ by the rule $l_a(x) := ax$. Prove that the map $G \to S(G) : a \mapsto l_a$ is an injective homomorphism of groups. [You may assume without proof that $l_a \in S(G)$ for all a]. ***(4 marks)***

(iii) (a) For a group G define its group of automorphisms $\mathrm{Aut}(G)$ and the group of inner automorphisms $\mathrm{Inn}(G)$. You should carefully define all the terms and notation used. ***(3 marks)***

(b) Prove that $\mathrm{Inn}(G)$ is a subgroup of $\mathrm{Aut}(G)$. ***(7 marks)***

(c) Determine $\mathrm{Aut}(\mathbf{Z}/24\mathbf{Z})$ and $\mathrm{Inn}(\mathbf{Z}/24\mathbf{Z})$ explaining your reasoning. Express $\mathrm{Aut}(\mathbf{Z}/24\mathbf{Z})$ as a direct product of cyclic groups of prime power order. ***(5 marks)***

2 (i) Define the centre of a group and prove that it is a normal subgroup. ***(4 marks)***

(ii) (a) Let H be a subgroup of G. Prove that $H \cap Z(G)$ is a subgroup of $Z(H)$. ***(2 marks)***

(b) Give an example of H and G where $Z(H) \neq H \cap Z(G)$. ***(2 marks)***

(iii) (a) Define the orthogonal group O_2 and the elements R_θ, S_θ of O_2. ***(3 marks)***

(b) By multiplying out matrices (and quoting relevant trigonometric identities) show that $R_\theta S_\phi = S_{\theta+\phi}$. ***(2 marks)***

For the following parts you may assume that all elements of O_2 are of the form R_θ or S_θ for suitable θ, and you may use the identities $R_\theta^{-1} = R_{-\theta}$, $S_\theta S_\phi = R_{\theta-\phi}$ and $S_\theta R_\phi = S_{\theta-\phi}$ without proving them.

(c) Determine the conjugacy class of S_θ in O_2. ***(2 marks)***

(d) The conjugacy class of R_θ is given by $\{R_\theta, R_\theta^{-1}\}$ (no proof required for this fact). By considering the conjugacy classes of its elements determine the centre of O_2. ***(3 marks)***

(iv) Let $T_4(\mathbf{Q})$ be the group of invertible 4×4 lower triangular matrices over $\mathbf{Q}$, i.e., matrices of the form

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \in \mathrm{GL}_4(\mathbf{Q}).$$

Prove that its centre is the set of scalar matrices $\{rE \mid r \in \mathbf{Q}^*\}$, where E is the 4×4 identity matrix. ***(7 marks)***

3 (i) (a) Give the definition of the action of a group G on a set X.
(3 marks)

(b) Given a homomorphism $\phi : G \to S(X)$ explain how to define an action of G on X and prove that it satisfies the necessary axioms.
(4 marks)

(ii) Let S be a subset of a group G. For $g \in G$ define

$$g * S := gSg^{-1} = \{gsg^{-1} : s \in S\}.$$

(a) Show that this defines a group action of G on its set of subsets.
(4 marks)

(b) If H is a subgroup of G prove that H is a normal subgroup of the stabilizer of H under this action, called the *normalizer* $N_G(H)$.
(4 marks)

(iii) Let C be a cube centred at the origin in $\mathbf{R}^3$ and write

$$H = \mathrm{Dir}(C) = \{A \in \mathrm{SO}_3 | AC = C\}.$$

(a) Describe a set of four things on which H acts non-trivially, and explain carefully how this gives a homomorphism $\phi : H \to S_4$.
(4 marks)

(b) Let x be a point on the surface of C that lies on an edge, close to a corner but not at the corner. Prove that the orbit of x has 24 elements. By considering the orders of various sets, and assuming that ϕ is injective, deduce that ϕ is surjective. ***(6 marks)***

4 (i) State the Sylow theorems. You should carefully define all the terms and notation used. ***(5 marks)***

(ii) Determine the number of Sylow 5-subgroups of S_5. ***(5 marks)***

(iii) Let G be a group of order 99.

(a) Show that G has a normal subgroup N of order 11. ***(4 marks)***

(b) Prove that if P is a Sylow 3-subgroup of G then every element of P commutes with every element of N. ***(7 marks)***

(c) Deduce that G is abelian. [You may use without proof the fact that for every prime p a group of order p^2 is abelian.] ***(4 marks)***

End of Question Paper