The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2013–2014**

MAS377 Mathematical Biology **2 hours**

Marks will be awarded for your best ***four*** *answers.*

1 A gardener wants to control the number of aphids in their garden by releasing ladybirds, a natural predator of aphids. The dynamics of aphids (A) and ladybirds (L) are given by the following ordinary differential equations,

$$\frac{dA}{dt} = A(a - bA - cL) \quad (1)$$

$$\frac{dL}{dt} = L(ecA - d). \quad (2)$$

where a, b, c, d, e are positive constants.

(i) Give a biological interpretation of the parameters c and e. ***(4 marks)***

(ii) Find the two non-trivial equilibria of this system. ***(6 marks)***

(iii) Calculate the Jacobian, J, of the system at the general equilibrium (A_*, L_*). Substitute in the two equilibria from part (ii), and find a condition on c to determine which equilibria is stable. ***(10 marks)***

(iv) Show that whenever the ladybird stably co-exists with the aphids, it successfully reduces aphid numbers. ***(5 marks)***

2 Suppose a human population is exposed to an infectious disease, with the dynamics of this system given by the following ordinary differential equations,

$$\frac{dS}{dt} = \mu(N - vR) - \beta SI - \mu S \tag{3}$$

$$\frac{dI}{dt} = \beta SI - (\mu + \gamma)I \tag{4}$$

$$\frac{dR}{dt} = \gamma I - \mu R + v\mu R. \tag{5}$$

where individuals reproduce and die at rate μ, the transmission rate has coefficient β and infected individuals recover at rate γ. All parameters are positive constants with $0 < v < 1$.

(i) What biological process does the parameter v represent? ***(2 marks)***

(ii) Explain why this system can be fully described by just the first two ordinary differential equations, dS/dt and dI/dt. Write out this reduced model, using a suitable substitution for R wherever necessary. ***(5 marks)***

(iii) State the R_0 value for this model (you need not derive it) and explain what it represents. ***(3 marks)***

(iv) Sketch a phase portrait for this system showing an endemic equilibrium, assuming that $R_0 > 1$ and $0 < v < 1$. Your plot should show the bounds of the biologically-feasible region, nullclines, qualitative directions of flow and at least one sample trajectory. ***(10 marks)***

(v) Considering the region in your phase portrait in which the $dS/dt = 0$ nullcline is within the biologically-feasible region, show that there is always an endemic equilibrium for any $0 < v < 1$ when $R_0 > 1$. ***(5 marks)***

3 The concentration of a messenger RNA $m(t)$ can be modelled as

$$\frac{dm}{dt} = -\mu m + f(t),$$

where μ is a positive constant.

(i) Suggest biological meanings for μ and $f(t)$. ***(2 marks)***

(ii) If

$$f(t) = \begin{cases} 0 & t < 0 \\ \alpha t & 0 < t \leq T \\ \alpha(2T - t) & T < t \leq 2T \\ 0 & 2T < t \end{cases}$$

and $m(0) = 0$, find $m(t)$ for $0 < t \leq T$. Show that, for small t, the mRNA concentration is approximately

$$m(t) \approx \frac{\alpha t^2}{2}.$$

(8 marks)

(iii) If $\mu T = 1$, find $m(t)$ for $T < t \leq 2T$. Show that the mRNA profile reaches a maximum value when $t \approx 1.5T$. ***(10 marks)***

(iv) Find $m(t)$ for $t > 2T$, and sketch the time-profile of $m(t)$. ***(5 marks)***

4 A model of two mutually-repressive genes, X_1 and X_2, is given by

$$\begin{aligned} \frac{dX_1}{dt} &= -\mu X_1 + f(X_2) \\ \frac{dX_2}{dt} &= -\mu X_2 + f(X_1), \end{aligned} \qquad (6)$$

where μ is a positive constant, and $f(X) = \dfrac{\theta^m}{\theta^m + X^m}$, where θ and m are positive constants.

(i) Sketch $f(X)$ for $m > 1$ and describe how your sketch depends on the values of θ and m. ***(4 marks)***

(ii) Sketch the nullclines of the model and show that, for suitable values of θ and m, the model possesses three distinct steady states. ***(4 marks)***

(iii) Show that the Jacobian of the model at a steady state is of the form

$$J = \begin{pmatrix} -\mu & A \\ B & -\mu \end{pmatrix}$$

and determine A and B.
By considering the gradients of the nullclines at each steady state, determine their stability. ***(12 marks)***

(iv) Sketch the phase portrait for the system when it has three steady states, showing representative trajectories. Given initial conditions

$$X_1(0) = \epsilon \ll \frac{1}{\mu}, \quad X_2(0) = 0,$$

sketch the form of the time-courses of X_1 and X_2. ***(5 marks)***

5 A non-dimensionalised reaction diffusion model is given by

$$\frac{\partial U}{\partial t} = a - U + U^2 V + \frac{\partial^2 U}{\partial x^2} \tag{7}$$

$$\frac{\partial V}{\partial t} = b - U^2 V + D\frac{\partial^2 V}{\partial x^2}, \tag{8}$$

where $U(x,t)$ and $V(x,t)$ are the concentrations of two chemicals at position x and time t, and a, b and D are positive parameters.

(i) Show that the model has a unique spatially uniform steady state (U_*, V_*) and find expressions for U_* and V_*. ***(4 marks)***

(ii) Show that (U_*, V_*) is stable to *spatially uniform* perturbations if and only if $(a+b)^3 > b - a$. ***(8 marks)***

(iii) By considering the growth rate of spatially periodic perturbations to (U_*, V_*), find three additional inequalities that the parameters must satisfy in order for diffusion-driven instability to occur. ***(13 marks)***

End of Question Paper