The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2013–2014**

MAS422 Magnetohydrodynamics **2 hours**

Answer all four questions.

1 (i) Consider a magnetic field with components

$$B_x(x,z) = \frac{\partial \psi}{\partial z}, \qquad B_y = B_y(x,z), \qquad B_z(x,z) = -\frac{\partial \psi}{\partial x},$$

where $\psi = \psi(x,z)$.

(a) Show that $\nabla \cdot \mathbf{B} = 0$. ***(2 marks)***

(b) Show that $\mathbf{B} \cdot \nabla \psi = 0$ and that projections of field lines in the xz-plane are given by ψ=constant. ***(6 marks)***

(c) Show that if

$$\mathbf{J} \times \mathbf{B} = \mathbf{0},$$

where $\mathbf{J}$ is the current density, then

$$B_y = B_y(\psi)$$

and ψ satisfies

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} + B_y \frac{dB_y}{d\psi} = 0.$$

(12 marks)

(ii) Calculate the approximate timescale (in years) for the decay of the interstellar magnetic field given the parameters $L = 3 \times 10^{16}$ m and $\eta = 3.6 \times 10^6$ cm^2 s^{-1}. ***(5 marks)***

2 (i) State the relationship between Lagrangian and Eulerian perturbations. Write the Lagrangian density perturbation in terms of the Eulerian perturbation. ***(4 marks)***

2 (continued)

(ii) Consider

$$\mathbf{B} = B_0(y/a, x/a, 0).$$

Sketch the field lines showing the direction. ***(4 marks)***

Calculate the magnetic tension and magnetic pressure forces. Comment on the direction of each. Give comments about the Lorentz force. ***(7 marks)***

(iii) Prove that

$$\frac{\partial \hat{\boldsymbol{\theta}}}{\partial \theta} = -\hat{\mathbf{r}},$$

where r and θ are two-dimensional polar coordinates with $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ being the corresponding unit vectors. ***(3 marks)***

(iv) Prove, using (iii), that for magnetic fields of the form $\mathbf{B} = B_\theta(r)\hat{\boldsymbol{\theta}}$,

$$(\mathbf{B}\cdot\nabla)\mathbf{B} = -\frac{B_\theta^2}{r}\hat{\mathbf{r}}.$$

(3 marks)

3 (i) (a) Ignoring viscosity, gravity and diffusivity, derive the linearised induction equation and equation of motion for adiabatic, ideal perturbation of the form $\sim \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}-\omega t}$, about a static and uniform equilibrium state with magnetic force alone. ***(6 marks)***

[You may use $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + (\nabla\cdot\mathbf{B})\mathbf{A} - (\nabla\cdot\mathbf{A})\mathbf{B}$ and $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$].

(b) Derive the dispersion relations for Alfvén and compressional Alfvén waves using linearised MHD equations obtained in (i). ***(8 marks)***

(c) State one property of each:
(a) Alfvén waves
(b) compressional Alfvén waves. ***(2 marks)***

(ii) (a) Write the induction equation for a perfectly conducting medium. Given a velocity field $\mathbf{v} = (yz, -xz, 0)$ and the initial condition on the magnetic field

$$\mathbf{B}(\mathbf{x}, 0) = (x, -y, 0),$$

find $\mathbf{B}(\mathbf{x}, t)$ by obtaining the Lagrangian coordinates corresponding to $\mathbf{v}$ and applying the Cauchy solution. ***(19 marks)***

 Question 3 continued on next page

3 (continued)

Hint: Use an initial condition $\mathbf{x}(0) = (a_1, a_2, a_3)$ and also make use of $\mathbf{x}(0)$ for the initial magnetic field $\mathbf{B}(\mathbf{x}, 0) = \mathbf{B}_0$.

(b) Having obtained $\mathbf{B}(\mathbf{x}, t)$ in (iv), verify by direct substitution that it is indeed the solution of the induction equation. ***(4 marks)***

4 (i) Consider a cartesian coordinate system where the magnetic and velocity fields are independent of y, i.e.

$$\mathbf{B} = \left(-\frac{\partial A}{\partial z}, B, \frac{\partial A}{\partial x}\right),$$

$$\mathbf{v} = \left(-\frac{\partial \psi}{\partial z}, v_y, \frac{\partial \psi}{\partial x}\right),$$

where A and ψ are scalar potential functions for magnetic and flow fields, respectively. Using the magnetic induction equation, the equations for the evolution of B and A can be given as

$$\frac{\partial B}{\partial t} + \left(\frac{\partial \psi}{\partial x}\frac{\partial B}{\partial z} - \frac{\partial \psi}{\partial z}\frac{\partial B}{\partial x}\right) = \left(\frac{\partial A}{\partial x}\frac{\partial v_y}{\partial z} - \frac{\partial A}{\partial z}\frac{\partial v_y}{\partial x}\right) - \left(\nabla \cdot (\alpha \nabla A) - \eta \nabla^2 B\right), \quad \text{(I)}$$

$$\frac{\partial A}{\partial t} + \left(\frac{\partial \psi}{\partial x}\frac{\partial A}{\partial z} - \frac{\partial \psi}{\partial z}\frac{\partial A}{\partial x}\right) = \alpha B + \eta \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] A. \quad \text{(II)}$$

Simplify the above equations (I and II) by setting $\psi = 0$, $v_y = Qz$ and ignoring the term containing α in the equation for the evolution of B. ***(2 marks)***

(ii) Assuming solutions of the form $(A, B) = (\tilde{A}, \tilde{B})e^{\sigma t - i(k_x x + k_z z)}$ in the simplified equations obtained in (i), show that the following condition is satisfied for the solutions to exist

$$(\sigma + \eta \kappa^2)^2 = -i k_x \alpha Q,$$

where $\kappa^2 = k_x^2 + k_z^2$. ***(6 marks)***

(iii) In the equations for the evolution of B and A in (i), assume $\psi = v_y = 0$ but retain the term containing α to be a constant. Substitute the same solutions of the form $(A, B) = (\tilde{A}, \tilde{B})e^{\sigma t - i(k_x x + k_z z)}$ and show that now the solution satisfies the relation

$$\sigma = \pm \alpha \kappa - \eta \kappa^2.$$

(7 marks)

End of Question Paper