The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring semester 2013-2014**

Advanced Operations Research **2 Hours**

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 (i) Use the two-phase method to solve the following linear programming problem:

$$\text{Min } z = 2x_1 + 3x_2 - 5x_3$$
$$x_1 + x_2 + x_3 = 7$$
$$2x_1 - 5x_2 + x_3 \geq 10$$

where x_1, x_2 and x_3 are non-negative. Clearly state your optimal solution and optimal cost. (Hint: Not counting the pre-processing steps, you should be able to finish phase 1 with two simplex iterations and phase 2 with one simplex iteration.) ***(25 marks)***

2 (i) Bill decides to watch at least 6 movies showing in cinemas in five nearby towns. If he travels to a town, he will stay there until he has watched all the movies. The following table provides the information about the 9 movies on show in each town:

Location	Movies	Round-trip miles	Cost per movie (£)
Town 1	1, 3, 4	0	7.95
Town 2	1, 6, 8	25	5.50
Town 3	2, 5, 6, 7	30	5.00
Town 4	1, 8, 9	28	7.00
Town 5	2, 4, 5, 7	40	4.95

The cost of driving is 75 pence per mile. Formulate the mixed integer/linear programming problem that determines the towns Bill needs to visit while minimising his total cost. Do NOT solve the resulting problem.

(13 marks)

(ii) A network of pipelines connecting five sites is shown in Figure 1. The flow capacities in both directions of each link are given in the figure. Formulate the linear programming problem that determines the maximal flow from site 1 to site 5. Do NOT solve the resulting problem. ***(12 marks)***

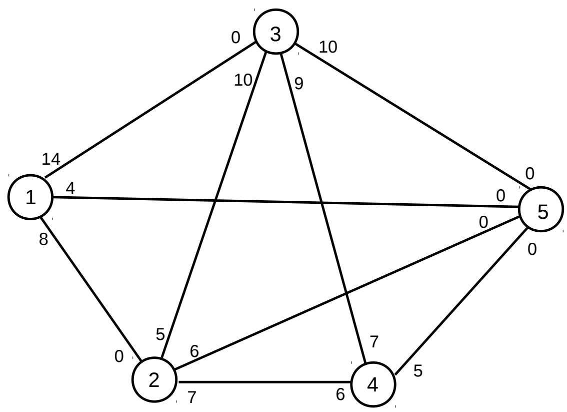

Figure 1: The network of pipelines in problem 2(ii).

3 A company produces units of product A, B and C, using raw materials M_1 and M_2 and labour. The requirements for each product are given in the following table:

	M_1(kg)	M_2(kg)	Labour(hour)	Profit (£)
A	3	8	2	69
B	6	12.5	3	111
C	2	3.5	1	42

The company has available in the next month 1 ton of M_1, 2.5 tons of M_2, and 550 hours of labour. To determine the optimal production schedule, the manager defines x_1, x_2 and x_3 to be the number of A's, B's, and C's to be produced in the next month, and formulates the following model for the profit:

$$\text{Maximise} \quad z = 69x_1 + 111x_2 + 42x_3$$

subject to

$$3x_1 + 6x_2 + 2x_3 \leq 1000$$

$$8x_1 + 12.5x_2 + 3.5x_3 \leq 2500$$

$$2x_1 + 3x_2 + x_3 \leq 550$$

where x_1, x_2, and x_3 are non-negative. Using slack variables x_4, x_5, and x_6, the optimal tableau is found as follows:

Basis	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	0	15	0	15	0	12	21600
x_3	0	3	1	2	0	-3	350
x_5	0	2	0	1	1	-5.5	475
x_1	1	0	0	-1	0	2	100

(i) What is the optimal production schedule and the corresponding profit? ***(2 marks)***

(ii) What are the optimal values of the dual variables? What do they tell us about the constraints? ***(3 marks)***

(iii) Suppose the amount of raw material M_1 used in B can be reduced from current 6kg/unit requirement, causing no other changes in the data in the problem. By how much does it need to be decreased to have B introduced into the optimal production schedule? ***(6 marks)***

(iv) The manager is contemplating manufacturing another product D. A unit of D requires 4kg of M_1, 5kg of M_2, and 1.5 hours of labour. What is the minimum profit for D so that it is profitable to be included in the optimal production schedule? ***(5 marks)***

(v) Denote the amount of available raw material M_1 by b_1. Find the range of b_1 for which the optimal production schedule remains unchanged. ***(7 marks)***

(vi) Suppose additional M_1 can be purchased at a cost of £7.75/kg. Should more be purchased and why? ***(2 marks)***

4 The payoff matrix for a two-person zero-sum game is given as follows:

$$A = \begin{bmatrix} 1 & -4 & 2 \\ -3 & 4 & -2 \\ -2 & 3 & 1 \\ 0 & -5 & -1 \end{bmatrix},$$

where the rows represent the pure strategies for player A and the columns represent those for player B.

(i) Show that the game has no pure strategy equilibrium solution.
(2 marks)

(ii) Set up the two linear programming problems from which you can solve the optimal mixed strategy solutions for both players. Do NOT solve the resulting problem, but briefly explain how the solutions are related to the optimal strategies and the value of the game. ***(5 marks)***

(iii) By removing dominated pure strategies, reduce the payoff matrix to a 3×2 matrix. ***(3 marks)***

(iv) Using the reduced payoff matrix, find the optimal strategies for the players and the value of the game (you may use graphs to assist your calculation).
(15 marks)

5 (i) For the following maximisation primal linear programming problem

$$\max \ z(x) = c^T x, \quad \text{subject to} \quad Ax \le b, x \ge 0, b \ge 0,$$

define its Lagrangian dual function, and show that the dual problem is

$$\min \ v(y) = b^T y, \quad \text{subject to} \quad A^T y \ge c, y \ge 0.$$

(7 marks)

(ii) Let B be the optimal basis for the primal problem in its canonical form. Show that $y^* = B^{-T} c_B$ satisfies the constraints of the dual problem, i.e., $A^T y^* \ge c$, and that y^* equals the shadow costs of the constraints of the primal problem. ***(6 marks)***

(iii) Write down the complementary slackness conditions for the above primal and dual pair. Give an interpretation of the conditions in terms of shadow costs and reduced costs. ***(6 marks)***

(iv) For the following linear programming problem

$$\begin{aligned} \text{Maximise} \quad & z = x_2 + 2x_3 \\ \text{subject to} \quad & \\ & x_1 - 2x_2 + 2x_3 \le -8 \\ & -x_1 + x_2 + x_3 \le 5 \\ & 2x_1 - x_2 + 4x_3 \le 10 \\ & x_1, x_2, x_3 \ge 0, \end{aligned}$$

introducing slack variables x_4, x_5, $x_6 \ge 0$, use the dual simplex algorithm to show that $x_2 = 4$, $x_5 = 1$, $x_6 = 14$, $x_1 = x_3 = x_4 = 0$ is a feasible basic solution (Hint: you need only one dual simplex step). ***(6 marks)***

End of Question Paper