Data provided: Formula sheet

MAS424

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2013–2014**

Differential Equations (Advanced) **2 hours**

Attempt all questions.

1 A two-species predator-prey system with populations x and y respectively is modelled by the equations

$$\frac{dx}{dt} = Ax\left(1 - \frac{x}{K}\right) - Bxy(1 - e^{-Cx})$$

and

$$\frac{dy}{dt} = -Dy + Ey(1 - e^{-Cx}),$$

in which the six parameters (A, B, C, D, E, K) are strictly positive.

(i) What do each of the four terms on the righthand sides of these equations imply ecologically? ***(4 marks)***

(ii) Non-dimensionalise the system using the scalings

$$X = \frac{x}{K}, \quad Y = \frac{By}{A}, \quad T = At \quad \alpha = \frac{E}{A}, \quad \delta = \frac{D}{A}, \quad \beta = CK,$$

to obtain the non-dimensional system given by

$$\frac{dX}{dT} = X(1 - X) - XY(1 - e^{-\beta X}),$$

and

$$\frac{dY}{dT} = -\delta Y + \alpha Y(1 - e^{-\beta X}).$$

(4 marks)

(iii) Determine the co-ordinates of the three critical points, noting any parameter restrictions. ***(7 marks)***

(iv) Determine the stability of each of the critical points. ***(10 marks)***

2 It is dreamed that in the future, rapid travel between cities might occur via the use of tunnels bored directly through the earth. We model the earth as a sphere of uniform density, ρ, and radius R. You are given that the volume of a sphere is $\frac{4}{3}\pi r^3$ and that $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$.

(i) Consider that there exists a rectilinear tunnel that passes through the centre of the earth connecting the North Pole to the South Pole. At time t, a capsule of mass m is travelling in the tunnel and is at a distance r from the centre of the earth. The gravitational force on the capsule is $F(r) = -\frac{GmM(r)}{r^2}$, where G is Newton's gravitational constant and $M(r)$ is the mass of that portion of the earth that is not further from the centre than r. The net attraction of the shell of matter between r and R is zero, whilst that of the sphere of radius r acts as though all its mass were concentrated at the centre.

(a) Show that $F(r) = -\frac{4}{3}\pi\rho Gmr$. ***(2 marks)***

(b) At the beginning of the transit, the capsule starts from rest at the North Pole. Making use of Newton's second law of motion, derive the result

$$\dot{r}^2 = \frac{g}{R}(R^2 - r^2).$$

(7 marks)

(c) You are given that the earth's radius, $R = 6370000$ m, and the the acceleration due to gravity, $g = 9.81$ ms^{-1}. Calculate the total transit time for the capsule to travel from the North Pole to the South Pole. ***(7 marks)***

(ii) The Euler-Lagrange equation corresponding to a functional $f(u(\theta), u'(\theta))$ is

$$\frac{\partial F}{\partial u} - \frac{d}{d\theta}\left(\frac{\partial f}{\partial u'}\right) = 0.$$

(a) Show that

$$\frac{df}{d\theta} = \frac{d}{d\theta}\left[u'\frac{\partial f}{\partial u'}\right].$$

(4 marks)

(b) For the case of a capsule in transit along a curvilinear tunnel between two cities that are not diametrically opposite each other, it can be shown that an appropriate form for the functional, f, that arises in the problem of minimising transit time, is given by

$$f(u, u') = \sqrt{\frac{u^2 + u'^2}{1 - u^2}}.$$

Show that

$$u^4 = C^2(u^2 + u'^2)(1 - u^2),$$

where C is a constant of integration. ***(5 marks)***

3 (i) Consider the system of equations

$$\dot{x} = y + xF(r),$$
$$\dot{y} = -x + yF(r), \qquad (*)$$

where $r^2 = x^2 + y^2$.

(a) By making the substitutions $x = r\cos\theta$ and $y = r\sin\theta$ show that this system has a periodic solution for each value of r_0 such that $F(r_0) = 0$. ***(10 marks)***

(b) You are given that this periodic solution is a stable limit cycle in the case $F'(r_0) < 0$ and that it is an unstable limit cycle in the case $F'(r_0) > 0$.
For the case $F(r) = -(r-3)(r^2 - 6r + 5)$ find all the limit cycles and determine their stability. ***(7 marks)***

(ii) When the curve $y = y(x)$ that joins two points, (x_0, y_0) and (x_1, y_1), is rotated once about the $y-$axis, the surface generated is

$$S[y] = 2\pi \int_{x_0}^{x_1} y(1 + y'^2)^{\frac{1}{2}} \, dx.$$

Show that an extremal for this problem is given by

$$y(x) = p\cosh\left(\frac{x}{p} + q\right),$$

where p and q are constants. ***(8 marks)***

4 The one-dimensional heat equation can be written as

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2},$$

where the constant c^2 denotes the thermal diffusivity. This equation can be used to model the temperature distribution in a laterally insulated, homogeneous metal bar of length a.

(i) Use the technique of separating the variables, with $u(x,t) = F(x)G(x)$, to find the general solution of this equation that satisfies the boundary conditions

$$u(0,t) = 0\ ^\circ C, \quad u(a,t) = 0\ ^\circ C \quad \text{for all } \ t \geq 0.$$

(20 marks)

(ii) You are given that the initial temperature distribution along the bar is given by

$$u(x,0) = 100\left(\sin\frac{3\pi x}{80}\right)\ ^\circ C. \qquad (*)$$

If the length of the bar is 80 cm and $c^2 = 1.158$ cm^{2}s^{-1}, calculate how long it will take for the maximum temperature in the bar to drop to 50 $^\circ C$.

(5 marks)

End of Question Paper

List of Basic Formulae and Theorems

Theorem 1: If a periodic solution of the system of equations

$$\dot{x} = f(x, y), \qquad \dot{y} = g(x, y)$$

exists in a simply connected region, then $f_x + g_y = 0$ somewhere in that region.

Corollary: There are no periodic solutions in any simply connected region where $f_x + g_y \neq 0$ everywhere.

Theorem 2: The orbit $\mathcal{C}$ of a periodic solution must enclose at least one critical point.

Orthogonality conditions for trig functions

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0, \quad \int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0 \quad \text{when } m \neq n.$$

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0.$$

Extremals of functional

$$J[y] = \int_{x_0}^{x_1} f(y, y', x) \, dx$$

are the solutions to the Euler-Lagrange equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0.$$