The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2013–2014**

Algebraic Topology **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 In this question, your answers should explain why the statements are true or false, giving proofs or counterexamples as appropriate. You may quote results and calculations from lectures.

(i) Let $f, g : I \longrightarrow S^1$ be two maps from the unit interval $I = [0, 1]$ to the unit circle S^1 in $\mathbb{R}^2$ defined as follows:

$$f(x) = (\cos(2\pi x), \sin(2\pi x)),$$
$$g(x) = (1, 0).$$

(a) Is f homotopic to g? ***(2 marks)***

(b) Is f path homotopic to g? ***(2 marks)***

(ii) Is the torus T homotopy equivalent to the two dimensional sphere S^2? ***(4 marks)***

(iii) Is the unit interval $I = [0, 1]$ contractible? ***(4 marks)***

(iv) Calculate $\pi_1(\mathbb{R}P^2 \times S^1)$, where $\mathbb{R}P^2$ is two dimensional real projective space and S^1 is the circle. ***(4 marks)***

(v) Is π_1 of a space always abelian? ***(4 marks)***

2 As previously we use the notation S^1 for the circle and S^2 for the two dimensional sphere. The wedge sum of S^1 and S^2 is denoted $S^1 \vee S^2$.

You may use results from the lectures, but you should state these carefully.

(i) (a) Calculate $\pi_1(S^1 \vee S^2)$. ***(5 marks)***

(b) What is the universal cover for each of the spaces S^1, S^2 and $S^1 \vee S^2$? ***(3 marks)***

(c) Characterise all connected covers of $S^1 \vee S^2$ and prove that what you listed is the full characterisation. ***(6 marks)***

(ii) Construct a space with $\pi_1 = \langle a, b, c | abca^{-1}, c^2 \rangle$ and prove that your space's π_1 is as required. ***(6 marks)***

3 (i) Let C and D be chain complexes of abelian groups and let f and g be chain maps from C to D.

(a) Define what it means for P to be a chain homotopy between the chain maps $f, g : C \to D$. ***(2 marks)***

(b) Prove that chain homotopic maps induce the same homomorphism $H_n(C) \to H_n(D)$ for all n. ***(4 marks)***

(ii) Calculate all the homology groups of the chain complex C:

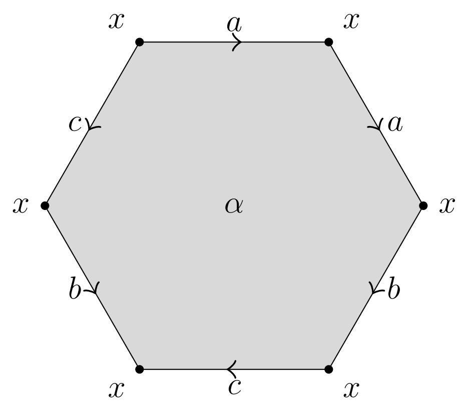
$$0 \xrightarrow{\delta_4} \mathbb{Z}\{a\} \xrightarrow{\delta_3} \mathbb{Z}\{b\} \oplus \mathbb{Z}\{c\} \xrightarrow{\delta_2} \mathbb{Z}\{d\} \oplus \mathbb{Z}\{e\} \xrightarrow{\delta_1} \mathbb{Z}\{f\} \xrightarrow{\delta_0} 0$$

where $C_n = 0$ for $n \geq 4$ and

$$\delta_1(d) = f, \quad \delta_1(e) = -f,$$
$$\delta_2(b) = 2d + 2e, \quad \delta_2(c) = 4d + 4e,$$
$$\delta_3(a) = 7c - 14b.$$

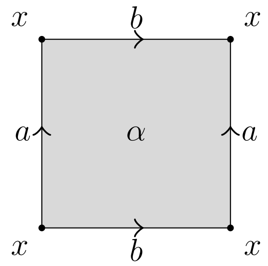
(The notation $\mathbb{Z}\{x\}$ means the free abelian group on generator x.)
(8 marks)

(iii) Consider the cell complex indicated in the diagram below.



Write down the chain complex associated to this cell complex and calculate all its homology groups. ***(6 marks)***

4 (i) Consider the cell complex X as pictured below.



Let A be the subcomplex of X consisting of the 0-cell x and the 1-cell a.

(a) Write down the chain complexes of X and A and the relative chain complex $C(X, A)$. ***(3 marks)***

(b) Calculate the relative homology groups $H_n(X, A)$, for $n \geq 0$. ***(2 marks)***

(c) Describe the quotient space X/A. ***(2 marks)***

(d) Write down the augmented chain complex of X/A and check that the reduced homology of X/A is the same as the relative homology calculated in part (b). ***(2 marks)***

(ii) (a) Let X be a topological space, with subspaces A and B such that X is the union of the interiors of A and B. Write down the Mayer-Vietoris long exact sequence involving the homology groups of the spaces $A \cap B$, A, B and X. ***(3 marks)***

(b) Recall that the suspension of X, denoted SX, is the quotient space $X \times [0, 1]/\sim$, where the equivalence relation $\sim$ is given by

$$(x_1, 0) \sim (x_2, 0) \quad \text{for all } x_1, x_2 \in X, \quad \text{and}$$
$$(x_1, 1) \sim (x_2, 1) \quad \text{for all } x_1, x_2 \in X.$$

Use the Mayer-Vietoris sequence to show that $\widetilde{H}_n(SX) \cong \widetilde{H}_{n-1}(X)$ for all $n \geq 1$. [**Hint:** cover the suspension with two "cones".] ***(8 marks)***

End of Question Paper