The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2013–2014

MAS436: Functional Analysis — **2 hours 30 minutes**

Answer ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

Throughout this paper, unless otherwise stated, all normed vector spaces are either over the field of real numbers, $\mathbb{R}$, or the field of complex numbers, $\mathbb{C}$

1 (i) Let V and W be normed vector spaces. State the definition of a *bounded linear map* between V and W. Define the *norm* of a bounded linear map. ***(4 marks)***

(ii) Let $Hom(V, W)$ be the vector space of bounded linear maps from V to W. Show that taking the norm of a bounded linear map makes $Hom(V, W)$ into a normed vector space. ***(4 marks)***

(iii) State the definition of a Banach space. ***(2 marks)***

(iv) Show that if V is a normed vector space and W is a Banach space, then the space $Hom(V, W)$ is also a Banach space. ***(10 marks)***

(v) Let V be a normed vector space. Define the dual space, V^*. ***(2 marks)***

(vi) Precisely under what circumstances is the dual space V^* complete? Justify your answer. ***(3 marks)***

2 (i) (a) State the open mapping theorem. ***(3 marks)***

(b) Let V and W be Banach spaces, and let $T\colon V \to W$ be a bijective bounded linear map. Is $T^{-1}\colon W \to V$ necessarily bounded? Justify your answer.
(3 marks)

(ii) Let P be the vector space of all real polynomial functions.

(a) Show that we can define a norm on P by the formula

$$\|a_0 + a_1x + a_2x^2 + \cdots + a_nx^n\| = |a_0| + |a_1| + \cdots + |a_n|$$

(5 marks)

(b) Define a linear map $S\colon P \to P$ by the formula

$$S(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) = a_0 + \frac{1}{2}a_1x + \frac{1}{3}a_2x^2 + \cdots + \frac{1}{n+1}a_nx^n$$

Is the map S bounded? Justify your answer. ***(3 marks)***

(c) Write down a formula for S^{-1}. Is S^{-1} a bounded linear map?
(6 marks)

(iii) Is P a Banach space? Justify your answer. ***(4 marks)***

3 (i) Let H be a Hilbert space, and let $T\colon H \to H$ be a bounded linear map. State the definition of the adjoint, T^*, of T. ***(3 marks)***

(ii) Let $y \in H$. Show that

$$\|y\| = \sup\{|\langle x, y\rangle| \mid x \in H,\ \|x\| \leq 1\}.$$

(6 marks)

[If desired, you may use the Cauchy-Schwarz inequality without proof.]

(iii) Let $T\colon H \to H$ be a bounded linear map. Show that the adjoint, $T^*\colon H \to H$ is a bounded linear map, with norm $\|T^*\| = \|T\|$. ***(8 marks)***

(iv) Define bounded linear maps $A, B, C\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by the formulae

$$(Af)(x) = f(x+1) \qquad (Bf)(x) = \int_{-\infty}^{x} e^{-t^2} f(t)\, dt \qquad (Cf)(x) = \int_{-\infty}^{x} e^{-t^2} f(t+1)\, dt$$

respectively.

Compute A^*, B^* and C^*. ***(8 marks)***

4 (i) Let A be a unital Banach algebra over $\mathbb{C}$.

(a) Define the *spectrum* of an element $x \in A$. ***(3 marks)***

(b) State the spectral mapping theorem for polynomials. ***(2 marks)***

(ii) Equip the space $\mathbb{C}^n$ with the norm

$$\|(x_1, \dots, x_n)\| = |x_1| + \cdots + |x_n|$$

(a) Show that every linear map $T \colon \mathbb{C}^n \to \mathbb{C}^n$ is bounded. ***(5 marks)***

(b) Show that if $T \colon \mathbb{C}^n \to \mathbb{C}^n$ is a bounded linear map, then the spectrum of T is the set of eigenvalues of T. ***(4 marks)***

(iii) Determine the spectrum of each of the following operators from $L^2[0,1]$ to $L^2[0,1]$:

$$R(f)(x) = xf(x) \qquad S(f)(x) = (x^2+2)f(x)$$

(11 marks)

5 (i) State what is meant when we say a subset of a normed vector space is compact. ***(3 marks)***

(ii) Let V and W be normed vector spaces. State the definition of a compact operator between V and W. ***(3 marks)***

(iii) Prove that any bounded linear map with finite-dimensional image is compact. You may use without proof the fact that a closed bounded subset of a finite-dimensional normed vector space is compact. ***(4 marks)***

(iv) Prove that if $T \colon V \to V$ is a bounded linear operator, and $K \colon V \to V$ is a compact operator, then the composite $TK \colon V \to V$ is compact. ***(5 marks)***

(v) Write down an example of a linear map which is *not* compact. Why not? ***(5 marks)***

(vi) Prove that the linear map $S \colon \ell^2 \to \ell^2$ defined by the formula

$$S(a_1, a_2, a_3, \dots) = \left(a_1, \frac{a_2}{2}, \frac{a_3}{3}, \dots\right)$$

is compact. You may use without proof the fact that the norm-limit of a sequence of compact linear maps is also compact. ***(5 marks)***

End of Question Paper