2 hours 30 minutes

The University Of Sheffield.

MAS 441

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2013–2014**

Optics and Symplectic Geometry **2 hours 30 minutes**

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

Throughout the paper I denotes an identity matrix and J denotes a matrix of the form $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. All matrices have real entries. The standard symplectic form Ω on $\mathbb{R}^{2n}$ is defined by $\Omega(Z, Z') = Q \centerdot P' - P \centerdot Q'$, where $Z = (Q, P)$ and $Z' = (Q', P')$ are elements of $\mathbb{R}^{2n}$.

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Do not remove it from the hall

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1 (i) Define what it means for (V, ω) to be a *symplectic vector space.*

Define what it means for a vector subspace $W \subseteq V$ to be a *symplectic subspace of* (V, ω).

Define the *symplectic perp* or *skew* $W^\wedge$ *of* W. **(5 marks)**

(ii) Let (V, ω) be a symplectic vector space of dimension $2n \geqslant 2$. Prove that there is a subspace $W \subseteq V$ of dimension 2 such that $W^\wedge$ is symplectic and $V = W \oplus W^\wedge$. **(12 marks)**

(iii) Let (V, ω) be a symplectic vector space of dimension $2n$ and let $W \subseteq V$ be a vector subspace. Using the fact that $\dim W^\wedge = 2n - \dim W$ or otherwise, prove that $(W^\wedge)^\wedge = W$. **(3 marks)**

(iv) Let (V, ω) be a symplectic vector space and let $W \subseteq V$ be a vector subspace.

Show that W is a symplectic subspace if and only if $W \cap W^\wedge = \{0\}$. **(2 marks)**

(v) Let (V, ω) be a symplectic vector space and let $W \subseteq V$ be a vector subspace.

Is it true that if W is a symplectic subspace, then $W^\wedge$ is also? Either prove this, or exhibit a counterexample. **(3 marks)**

2 Let $\widetilde{\mathscr{L}_2}$ denote the set of all oriented straight lines in the plane $\mathbb{R}^2$.

(a) Define, using sketch diagrams if you wish, a bijective map from $\widetilde{\mathscr{L}_2}$ to $U \times \mathbb{R}$, where U is a unit circle in the plane.

Describe carefully the map $U \times \mathbb{R} \to \widetilde{\mathscr{L}_2}$ which is inverse to your map. **(6 marks)**

(b) Describe carefully a bijective map $\widetilde{\mathscr{L}_2} \to M$ where

$$M := \{(X, Y) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid |X|^2 = 1,\ X \boldsymbol{.} Y = 0\}$$

and its inverse map $M \to \widetilde{\mathscr{L}_2}$. **(3 marks)**

(c) Fix a point $(X, Y) \in M$ and consider a smooth curve in M given by $(X(t), Y(t))$ for $t \in \mathbb{R}$ with $X(0) = X,\ Y(0) = Y$.

Writing $Q = \dot{X}(0),\ P = \dot{Y}(0)$, show that

$$Q \boldsymbol{.} X = 0, \qquad Q \boldsymbol{.} Y + X \boldsymbol{.} P = 0. \qquad (*)$$

Conversely show that every $(Q, P) \in \mathbb{R}^2 \times \mathbb{R}^2$ which satisfies the conditions $(*)$ is the derivative of a curve at (X, Y) **(10 marks)**

(d) Writing

$$T_{(X,Y)} = \{(Q, P) \mid Q \boldsymbol{.} X = 0,\ Q \boldsymbol{.} Y + X \boldsymbol{.} P = 0\},$$

show that $T_{(X,Y)}$ is a symplectic subspace of $\mathbb{R}^2 \times \mathbb{R}^2$ with respect to Ω. **(6 marks)**

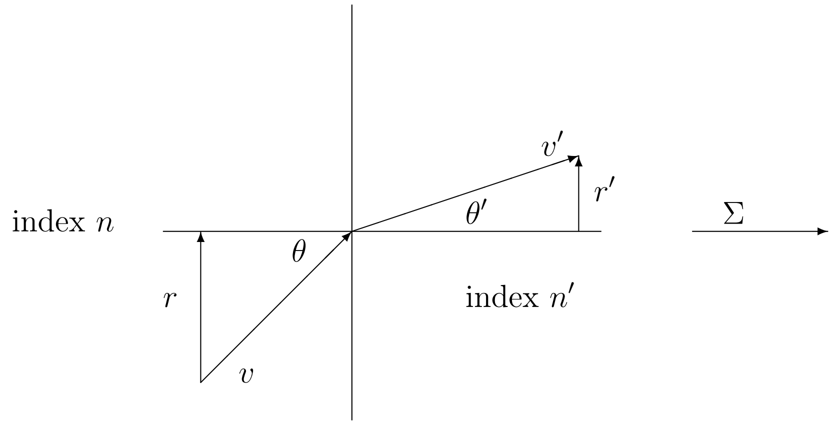

Figure 1: For Question **3**(i). The central vertical line represents a plane in $\mathbb{R}^3$.

3 (i) Figure 1 shows two unit vectors v, v' in $\mathbb{R}^3$ and a third unit vector Σ which is normal to the vertical plane separating the region of index of refraction n from the region of index of refraction n'.

Starting from Snell's Law in the form $n \sin\theta = n' \sin\theta'$, and referring to the situation in Figure 1, obtain a vector equation expressing Snell's Law in terms of v, v', n, n' and Σ. **(8 marks)**

(ii) Let $S = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ be a $2n \times 2n$ matrix in block form, where A, B, C and D denote $n \times n$ matrices.

(a) Prove that S is symplectic if and only if the three equations

$$A^{\mathsf{T}}C = C^{\mathsf{T}}A, \qquad B^{\mathsf{T}}D = D^{\mathsf{T}}B, \qquad A^{\mathsf{T}}D - C^{\mathsf{T}}B = I,$$

hold.

(b) Let W be an n-dimensional subspace of $\mathbb{R}^{2n}$, and let $G_1, \ldots, G_n$ be a basis for W. Write the matrix which has $G_1, \ldots, G_n$ as its columns in block form as $\left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right]$.

Show that W is Lagrangian with respect to Ω if and only if $M^{\mathsf{T}}N$ is symmetric.

(c) Let $S = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ be a $2n \times 2n$ symplectic matrix. Find a simple condition on B or on D for the Lagrangian subspace corresponding to $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$ to be transversal to $0 \times \mathbb{R}^n$. **(13 marks)**

(iii) Let (V_1, ω_1) and (V_2, ω_2) be symplectic vector spaces. A *symplectic map* $\varphi\colon V_1 \to V_2$ is a linear map such that $\omega_2(\varphi(v), \varphi(v')) = \omega_1(v, v')$ for all $v, v' \in V_1$.

Show that a symplectic map $\varphi\colon V_1 \to V_2$ is injective. **(4 marks)**

4 (i) In linear optics the matrix which transforms variables (Q,P) at z to variables (Q',P') at z' is $\left[\begin{smallmatrix} I & wI \\ 0 & I \end{smallmatrix}\right]$ where $w = \dfrac{z'-z}{n}$ and n is the index of refraction.

Further, given a paraboloid boundary $z = z_0 + \frac{1}{2}(fq_1^2 + 2gq_1q_2 + hq_2^2)$ between media with indexes of refraction n and n', the matrix which transforms variables (Q,P) to variables (Q',P') is $\left[\begin{smallmatrix} I & 0 \\ M & I \end{smallmatrix}\right]$ where $M = (n'-n)\left[\begin{smallmatrix} f & g \\ g & h \end{smallmatrix}\right]$.

(a) Use these facts to find the matrix which describes the following situation: two media of refractive indexes n (to the left) and n' (to the right) are separated by a surface given by

$$z = z_0 + \tfrac{1}{2}(fq_1^2 + 2gq_1q_2 + hq_2^2),$$

the variables (Q,P) are at $z = z_0 - k$, and the variables (Q',P') are at $z = z_0 + k'$ where $k, k' > 0$. **(5 marks)**

(b) Consider rays which are parallel to the z–axis at $z = z_0 - k$, and write $w' = \frac{k'}{n'}$.

In each case below describe in a few words the light which emerges at $z = z_0 + k'$:

- $-\dfrac{1}{w'}$ is not an eigenvalue of M;
- $-\dfrac{1}{w'}$ is an eigenvalue of M and the other eigenvalue is distinct;
- $-\dfrac{1}{w'}$ is a double eigenvalue of M.

(8 marks)

(ii) Let M be an $n \times n$ matrix, and write

$$W = \{(Q,P) \mid P = MQ\}.$$

When is L Lagrangian in $\mathbb{R}^{2n}$ with respect to Ω ? **(6 marks)**

(iii) Let A be a symmetric $n \times n$ matrix, let C be a symmetric $k \times k$ matrix, and let B be any $n \times k$ matrix. Prove that Ω is zero on

$$L = \{(Q, AQ + BX) \mid Q \in \mathbb{R}^n,\ X \in \mathbb{R}^k,\ B^{\mathsf{T}}Q + CX = 0\};$$

that is, show that $\Omega(Z,Z') = 0$ for $Z, Z' \in L$. **(6 marks)**

5 (i) Take $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Sp(2)$ with $b \neq 0$. Show that a choice of q and q' determine p and p' such that $\begin{bmatrix} q' \\ p' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix}$.

Define $\Gamma\colon \mathbb{R}^2 \to \mathbb{R}$ by $\Gamma(q, q') = \frac{1}{b}(qq' - \frac{1}{2}aq^2 - \frac{1}{2}dq'^2)$.

Show that

$$\frac{\partial \Gamma}{\partial q} = p, \qquad \frac{\partial \Gamma}{\partial q'} = -p'.$$

(4 marks)

(ii) Define when a smooth map $\varphi\colon \mathbb{R}^4 \to \mathbb{R}^4$ is a *symplectomorphism.*

Writing $\varphi(Q, P) = (F(Q, P), G(Q, P))$, formulate your definition in terms of the partial derivative matrices of φ. You may use the result of Question 3(ii)(a) without proof. **(5 marks)**

(iii) Let $\varphi\colon \mathbb{R}^4 \to \mathbb{R}^4$ be a symplectomorphism and write $\varphi(Q, P) = (F(Q, P), G(Q, P))$ as above. Assume that $\det\left(\frac{\partial F}{\partial P}\right) \neq 0$.

(a) The Local Diffeomorphism Theorem implies that it is possible to solve the equation $Q' = F(Q, P)$ for P in terms of Q and Q'. That is, there is a smooth function $H(Q, Q')$, defined in some open set, such that

$$Q' = F(Q, P) \quad \text{if and only if} \quad P = H(Q, Q'). \qquad (*)$$

Assuming this, show that

$$\frac{\partial F}{\partial Q} + \frac{\partial F}{\partial P}\frac{\partial H}{\partial Q} = 0, \qquad \frac{\partial F}{\partial P}\frac{\partial H}{\partial Q'} = I. \qquad (**)$$

(b) Now define $K(Q, Q') = -G(Q, H(Q, Q'))$. It follows that

$$\frac{\partial K}{\partial Q} = -\frac{\partial G}{\partial Q} - \frac{\partial G}{\partial P}\frac{\partial H}{\partial Q}, \qquad \frac{\partial K}{\partial Q'} = -\frac{\partial G}{\partial P}\frac{\partial H}{\partial Q'}. \qquad (***)$$

Using these, show that

$$\left(\frac{\partial K}{\partial Q}\right)^{\mathsf{T}} = \frac{\partial H}{\partial Q'}.$$

(16 marks)

End of Question Paper