Data provided: Formulae sheet

MAS445

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS

Spring Semester 2013-2014

Mathematics (Numerical methods and vector spaces)

Two hours

Marks will be awarded for your best FOUR answers

1 Consider the following differential equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

and answer the following questions.

(i) Classify the differential equation as either elliptic, parabolic or hyperbolic. ***(1 mark)***

(ii) By selecting the correct finite difference schemes, and correct changes in notation, derive the following scheme for solving Equation (1).

$$u_{i,j+1} = ru_{i-1,j} + (1-2r)u_{i,j} + ru_{i+1,j}$$

(3 marks)

(iii) Using the finite difference scheme, as found in part (ii), solve the differential equation between $0 \leq x \leq 5, 0 \leq t \leq 1$ for

$$h = 1, \qquad k = \frac{1}{4}, \qquad c^2 = 1,$$

where h is the step size in x and k is the step size in time. The following boundary conditions

$$u(t,0) = 25, \qquad u(t,5) = 0, \qquad u(0,x) = 25 - x^2,$$

are also applied. ***(10 marks)***

(iv) Repeat the calculation with the following changes.

$$h = 1, \qquad k = \frac{1}{2}, \qquad c^2 = 4, \qquad 0 \leq t \leq 1.5.$$

(5 marks)

(v) Discuss your results from section (iv) in terms of their physicality and suggest improvements to the numerical scheme used. ***(6 marks)***

2 (i) Determine the L and U matrices for the following system

$$A\mathbf{x} = b, \qquad A = \begin{bmatrix} 5 & 3 & 4 \\ 3 & 4 & 7 \\ 4 & 3 & 9 \end{bmatrix}, \qquad b = \begin{bmatrix} 4 \\ 5 \\ 8 \end{bmatrix}.$$

(6 marks)

(ii) Determine L^{-1} and U^{-1}. *(4 marks)*

(iii) Hence find the column vector $\mathbf{x}$. *(4 marks)*

(iv) Find the LU decomposition for the following tri-diagonal matrix

$$M = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}.$$

(6 marks)

(v) Using your results from part (iv) show M^{-1} to be

$$M^{-1} = \frac{1}{144} \begin{bmatrix} 55 & -21 & 8 & -3 & 1 \\ -21 & 63 & -24 & 9 & -3 \\ 8 & -24 & 64 & -24 & 8 \\ -3 & 9 & -24 & 63 & -21 \\ 1 & -3 & 8 & -21 & 55 \end{bmatrix}.$$

(5 marks)

3 (i) Write down the 'natural conditions' for a a cubic spline *(2 marks)*

(ii) Briefly discuss the main differences between the 'natural conditions' and the 'fixed conditions' for cubic splines and the influence of the chosen conditions on the spline. *(4 marks)*

(iii) Determine the cubic splines between the following data points

x	0	$\pi/3$	$2\pi/3$	π
f(x)	0	$\sqrt{3}/2$	$\sqrt{3}/2$	0

(17 marks)

(iv) Determine the value of the cubic spline at $x = 5\pi/6$. *(2 marks)*

4 (i) Show that the following set is an orthonormal set in $\mathcal{C}^4$.

$$\begin{aligned}\phi_1 &= (1,0,1,0)/\sqrt{2}\\ \phi_2 &= (1,1,-1,-1)/2\\ \phi_3 &= (0,1,0,1)/\sqrt{2}\\ \phi_4 &= (1,-1,-1,1,)/2\end{aligned}$$

(5 marks)

(ii) Find c_i, d_i such that

$$\begin{aligned} f &= (2,-i,0,3i) = \sum_{i=1}^{4} c_i\phi_i \\ g &= (-i,3,1+i,1) = \sum_{i=1}^{4} d_i\phi_i \end{aligned}$$

(8 marks)

(iii) Verify that

$$\begin{aligned} ||f||^2 &= \sum |c_i|^2 \\ ||g||^2 &= \sum |d_i|^2 \\ (f,g) &= \sum c_i d_i^* \end{aligned}$$

(6 marks)

(iv) Find the best approximation to (1,2,3,4) using only the vectors ϕ_1, ϕ_2, ϕ_3 and verify that the error is orthogonal to this best approximation.

(6 marks)

5 (i) Digital signals $\{f[n]\}_0^2$ of length 3 are obtained by sampling a random signal $f(t)$ at intervals $T/2$, where $f(t)$ has autocorrelation function

$$R_f(\tau) = \sigma^2 Sa(\pi\tau/T)$$

Write down the correlation matrix R, and use it to derive the K-L basis.

(20 marks)

(ii) These signals are to be compressed using only one member of the K-L basis. Which member of the basis should be used, and what is the associated mean square error?

(5 marks)

6 Let V be the Hilbert space of complex functions having finite energy in the interval $(0, 2)$, where the inner product is given by

$$(f, g) = \int_0^2 f(t) g^*(t)\, dt.$$

(i) Write down the corresponding norm. ***(2 marks)***

(ii) Find $||f||$ where f is the vector (a) $e^{i\omega t}$, (b) t, (c) $\dfrac{1}{2 + it}$. ***(6 marks)***

(iii) Find $z_1 = \big(t, e^{i\omega t}\big)$, $z_2 = \big(e^{i\omega t}, t\big)$, $z_3 = \big(t, ie^{i\omega t}\big)$ and verify that $z_2 = z_1^*$, $z_3 = -iz_1$. ***(9 marks)***

(iv) Find the component of t along $e^{i\omega t}$; call it $h(t)$. Verify that $t - h(t)$ is orthogonal to $e^{i\omega t}$. ***(8 marks)***

End of Question Paper

Formulae Sheet

Notation:

$$U(x_i,\, t_j) \equiv U_{i\,j}$$

Forward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t} \approx \frac{U_{i,j+1} - U_{ij}}{\Delta t}$$

Backward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t} \approx \frac{U_{ij} - U_{i,j-1}}{\Delta t}$$

Central difference formula for $\partial U/\partial x$:

$$\frac{\partial U}{\partial x} \approx \frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x}$$

Central difference formula for $\partial^2 U/\partial x^2$:

$$\frac{\partial^2 U}{\partial x^2} \approx \frac{U_{i+1,j} - 2U_{ij} + U_{i-1,j}}{\Delta x^2}$$