The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **2013–14**

Stochastic Processes and Finance **3 hours**

Candidates may bring to the examination a calculator that conforms to University regulations. Full marks may be obtained by complete answers to five questions. All answers will be marked, but credit will be given only for the best five answers.

1 $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.

(a) (i) Give the conditions $\mathbb{P}$ must satisfy to be a probability measure.

(ii) What does it mean to say $A \in \mathcal{F}$ is null?

(iii) If $B \in \mathcal{F}$, $B \subset A$ and A is null, use the defining properties of $\mathbb{P}$ to show that B is null.

(iv) Let $\mathbb{Q}$ be another probability measure on $\mathcal{F}$. What is meant by saying that $\mathbb{P}$ and $\mathbb{Q}$ are equivalent?

(v) Define the indicator function of $A \in \mathcal{F}$.

(vi) Let Y be a simple random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. Explain what this means and define $\mathbb{E}Y$ for such a Y.

(10 marks)

(b) Suppose Ω is $\{a, b, c, d\}$ and $\mathbb{P}(a) = \mathbb{P}(b) = \mathbb{P}(c) = \mathbb{P}(d)$. Two random variables, X and Y take the values in the following table:

	a	b	c	d
X	0	1	1	2
Y	7	6	2	1

(i) Confirm that $\mathbb{E}_{\mathbb{P}}X = 1$ and find $\mathbb{E}_{\mathbb{P}}Y$.

(ii) A new probability measure on the subsets A of Ω is defined by

$$\mathbb{Q}(A) = \int 1_A X \, d\mathbb{P}.$$

Find $\mathbb{E}_{\mathbb{Q}}Y$.

(iii) Are $\mathbb{P}$ and $\mathbb{Q}$ equivalent measures? Explain your answer.

(iv) Show that $\mathcal{V} = \{\varnothing, \{a\}, \{a, c, d\}, \{c, d\}, \{b, c, d\}, \Omega\}$ is not a σ-algebra of subsets of Ω. Find the smallest σ-algebra that contains $\mathcal{V}$.

(10 marks)

2 (a) Let X be an integrable random variable defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{G}$ and $\mathcal{H}$ be σ-algebras with $\mathcal{H} \subseteq \mathcal{G} \subseteq \mathcal{F}$.

(i) Define $\mathbb{E}(X|\mathcal{G})$.

(ii) Show that $\mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H}) = \mathbb{E}(X|\mathcal{H})$.

(iii) A function f is called convex if it lies above its tangent at every point: so for every y there is a $b(y)$ such that

$$f(x) \geq f(y) + b(y)(x - y).$$

By letting $y = \mathbb{E}(X|\mathcal{G})$ and $x = X$, use this to prove Jensen's inequality that $\mathbb{E}(f(X)|\mathcal{G}) \geq f(\mathbb{E}(X|\mathcal{G}))$.

(10 marks)

(b) (i) Give a precise mathematical definition of what is meant by saying $(B(t), t \geq 0)$ is a Brownian motion.

(ii) Use these defining properties to show that $(\hat{B}(t), t \geq 0)$ is a Brownian motion where, for a fixed $a > 0$, $\hat{B}(t) = B(t + a) - B(t)$.

(iii) Show that $e^{\sqrt{2}B(t)-t}$ has the martingale property.

[You may use that if $Y \sim N(0, \sigma^2)$ then $\mathbb{E}(e^{uY}) = \exp(\sigma^2 u^2/2)$.]

(10 marks)

3 Throughout this question $(B(t), t \geq 0)$ is a Brownian motion and $(\mathcal{F}_t, t \geq 0)$ is its natural filtration.

(a) (i) Give conditions on F for the stochastic integral

$$I_T(F) = \int_0^T F(t)dB(t)$$

to be defined and to have a finite variance.

(ii) Give three properties of $(I_t(F), 0 \leq t \leq T)$.

(iii) Suppose $(F(t), t \geq 0)$ is given by:

$$F(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1 \\ B(1) & \text{if } 1 < t \leq 2 \\ B(2) & \text{if } 2 < t \leq 3 \\ 0 & \text{if } t > 3. \end{cases}$$

Obtain an expression for

$$\int_0^3 F(s)dB(s),$$

in terms of $B(1)$, $B(2)$ and $B(3)$.

(iv) For the F in (iii), find the variance of $\int_0^3 F(s)dB(s)$. ***(11 marks)***

(b) (i) Obtain the stochastic differential of $\sin(B(t))$.

(ii) Suppose that, for $j = 1, 2$,

$$M_j(t) = \int_0^t F_j(s)dB(s) + \int_0^t G_j(s)ds$$

where F_j and G_j are such that M_j is an Itô processes. In this context, state Itô's product formula.

(iii) Obtain the stochastic differential of $Y(t) = e^{t/2}\sin(B(t))$. Deduce that Y is a martingale.

(9 marks)

4 Consider a finite market model based on a probability space with filtration $(\mathcal{F}_n, n \in \mathcal{T})$ where $\mathcal{T} = \{0, 1, \ldots, T\}$. The market consists of a risky financial asset with unit price at time n of $S(n)$ and a risk-free bond with constant interest rate. A (European contingent) claim is an undertaking to pay X at time T, where X is $\mathcal{F}_T$ measurable.

(a) In this context, define the following:

(i) the discounted prices;

(ii) a martingale measure for the discounted prices;

(iii) a portfolio;

(iv) a self-financing portfolio;

(v) an arbitrage opportunity;

(vi) an attainable claim.

(7 marks)

(b) Give a brief account of the first and second fundamental theorems of asset pricing. Give a formula for the price process for the claim X in an arbitrage-free complete market.

(5 marks)

(c) In a trinomial model with no interest, so that $r = 0$, the returns

$$K(n) = \frac{S(n) - S(n-1)}{S(n-1)}, \; n = 1, 2, 3, \ldots T,$$

are i.i.d. random variables taking the values

$$\begin{array}{lll} u & \text{with probability} & p \\ n & \text{with probability} & q \\ -d & \text{with probability} & 1-p-q \end{array},$$

where $0 < p, q, p + q < 1$ and $-d < n < u$.

(i) Find the values for $S(1)$, in terms of $S(0), u, n,$ and d, and their probabilities.

(ii) Show that risk-neutral probabilities, p_*, q_* must satisfy

$$p_* \frac{u+d}{d} + q_* \frac{n+d}{d} = 1.$$

(iii) Find all risk-neutral probabilities if $u = 0.3, n = 0.1, d = 0.1$.
Is the market arbitrage-free and complete? Explain your answer.

(8 marks)

5 $(B(t), t \geq 0)$ is a Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathcal{F}_t, t \geq 0)$ is its natural filtration. The price of a risky stock at time t, $S(t)$ is modelled by

$$S(t) = \exp\left\{\sigma B(t) + \int_0^t g(s)ds - \frac{1}{2}\sigma^2 t\right\}$$

where $g(s)$ is not random. There is a bond which involves no risk and offers a continuously compounded interest at rate r, so that a deposit of A produces Ae^{rt} after time t has passed.

(a) Explain why, under $\mathbb{P}$, $\log S(t)$ has a normal distribution with mean

$$\int_0^t g(s)ds - \frac{1}{2}\sigma^2 t$$

and variance $\sigma^2 t$. ***(3 marks)***

(b) (i) Find

$$\frac{d}{dt}\left(\int_0^t g(s)ds - \frac{1}{2}\sigma^2 t\right).$$

(ii) Use Itô's formula to show that $dS = S(gdt + \sigma dB)$.

(7 marks)

(c) Find the differential of the discounted price $\tilde{S}(t) = e^{-rt}S(t)$ (in terms of dt and dB). ***(1 mark)***

(d) The original measure $\mathbb{P}$ can be changed to the equivalent probability measure $\mathbb{Q}$ where

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left\{\int_0^T F(s)dB(s) - \frac{1}{2}\int_0^T F(s)^2 ds\right\},$$

and Girsanov's theorem then tells us that (under conditions that can be assumed to hold here)

$$W(t) = B(t) - \int_0^t F(s)ds$$

is a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{Q})$.

Show that $\tilde{S}$ is a martingale under $\mathbb{Q}$ when $F(t) = (r - g(t))/\sigma$.

(3 marks)

(e) Let U be normally distributed with mean $-\sigma^2 T/2$ and variance $\sigma^2 T$. Deduce that, under $\mathbb{Q}$, $\log \tilde{S}(T)$ is distributed like U. ***(3 marks)***

(f) A trader is offered $f(S(T))$ at time T. Show that the fair (no-arbitrage) price for this offer at time zero can be written as $\mathbb{E}\left[e^{-rT}f(se^{U+rT})\right]$.

(3 marks)

6 (a) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $(\mathcal{F}_n, n \in \mathcal{T})$, where $\mathcal{T} = \{0, 1, \ldots, T\}$. Let $X = (X(n), n \in \mathcal{T})$ be the discounted pay-off process, adapted to the filtration $(\mathcal{F}_n, n \in \mathcal{T})$, associated with an American contingent claim (ACC) with terminal time T.

(i) What is an American contingent claim?

(ii) Suppose τ is a stopping time. Define stopping time.

(iii) Explain what it means to say that τ solves the optimal stopping problem for $X = (X(n), n \in \mathcal{T})$.

Interpret the defining equation in terms of the ACC.

Explain why is it natural to consider only stopping times in seeking the optimal time.

(8 marks)

(b) Suppose that the UK government issues a 50-year bond at 2% interest rate in 2014 with par value £100.

(i) Calculate a rational price that you might expect to be able to sell your bond for in 2015 if UK government 50-year bonds are then being issued at an interest rate of 1.5%.

(ii) Calculate a rational price that you might expect to be able to sell your bond for in 2016 if UK government 50-year bonds are then being issued at an interest rate of 2.5%.

(iii) In making these calculations, why is it important that in both cases the bond still has a long time to expiry?

(3 marks)

(c) The term structure of a bond is the family of random variables $(P(t, T), 0 \leq t \leq T, T \geq 0)$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $P(t, T)$ is the value at t of a bond that pays one unit at T. How are the forward rate $f(t, T)$ and the spot-rate $r(t)$ obtained from $P(t, T)$? ***(3 marks)***

(d) In the same framework as in (c), define the discount factor by

$$A(t) = \exp\left(\int_0^t r(s)ds\right)$$

and the discounted term structure by $\widetilde{P}(t, T) = A(t)^{-1}P(t, T)$. Assume that there is a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ such that $(\widetilde{P}(t, T), 0 \leq t \leq T)$ is a $\mathbb{Q}$-martingale for a filtration $(\mathcal{F}_t, 0 \leq t \leq T)$ for each $T > 0$, and that the spot-rate is adapted to the filtration. Deduce that

$$P(t, T) = \mathbb{E}_{\mathbb{Q}}\left(\exp\left\{-\int_t^T r(s)ds\right\}\middle| \mathcal{F}_t\right).$$

(6 marks)

End of Question Paper