



Answer **four** questions. If you answer more than four questions, only your best four will be counted.

Throughout this paper, unless otherwise stated, all normed vector spaces are either over the field of real numbers, \mathbb{R} , or the field of complex numbers, \mathbb{C}

1 (i) Let V and W be normed vector spaces. State the definition of a *bounded linear map* between V and W . Define the *norm* of a bounded linear map. (4 marks)

(ii) Let $\text{Hom}(V, W)$ be the vector space of bounded linear maps from V to W . Show that taking the norm of a bounded linear map makes $\text{Hom}(V, W)$ into a normed vector space. (4 marks)

(iii) State the definition of a Banach space. (2 marks)

(iv) Show that if V is a normed vector space and W is a Banach space, then the space $\text{Hom}(V, W)$ is also a Banach space. (10 marks)

(v) Let V be a normed vector space. Define the dual space, V^* . (2 marks)

(vi) Precisely under what circumstances is the dual space V^* complete? Justify your answer. (3 marks)

2 (i) (a) State the open mapping theorem. (3 marks)

(b) Let V and W be Banach spaces, and let $T: V \rightarrow W$ be a bijective bounded linear map. Is $T^{-1}: W \rightarrow V$ necessarily bounded? Justify your answer.

(3 marks)

(ii) Let P be the vector space of all real polynomial functions.

(a) Show that we can define a norm on P by the formula

$$\|a_0 + a_1x + a_2x^2 + \cdots + a_nx^n\| = |a_0| + |a_1| + \cdots + |a_n|$$

(5 marks)

(b) Define a linear map $S: P \rightarrow P$ by the formula

$$S(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) = a_0 + \frac{1}{2}a_1x + \frac{1}{3}a_2x^2 + \cdots + \frac{1}{n+1}a_nx^n$$

Is the map S bounded? Justify your answer.

(3 marks)

(c) Write down a formula for S^{-1} . Is S^{-1} a bounded linear map?

(6 marks)

(iii) Is P a Banach space? Justify your answer.

(4 marks)

3 (i) Let H be a Hilbert space, and let $T: H \rightarrow H$ be a bounded linear map. State the definition of the adjoint, T^* , of T . (3 marks)

(ii) Let $y \in H$. Show that

$$\|y\| = \sup\{|\langle x, y \rangle| \mid x \in H, \|x\| \leq 1\}.$$

(6 marks)

[If desired, you may use the Cauchy-Schwarz inequality without proof.]

(iii) Let $T: H \rightarrow H$ be a bounded linear map. Show that the adjoint, $T^*: H \rightarrow H$ is a bounded linear map, with norm $\|T^*\| = \|T\|$. (8 marks)

(iv) Define bounded linear maps $A, B, C: L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ by the formulae

$$(Af)(x) = f(x+1) \quad (Bf)(x) = \int_{-\infty}^x e^{-t^2} f(t) dt \quad (Cf)(x) = \int_{-\infty}^x e^{-t^2} f(t+1) dt$$

respectively.

Compute A^* , B^* and C^* .

(8 marks)

- 4 (i) Let A be a unital Banach algebra over \mathbb{C} .
- (a) Define the *spectrum* of an element $x \in A$. (3 marks)
- (b) State the spectral mapping theorem for polynomials. (2 marks)
- (ii) Equip the space \mathbb{C}^n with the norm

$$\|(x_1, \dots, x_n)\| = |x_1| + \dots + |x_n|$$

- (a) Show that every linear map $T: \mathbb{C}^n \rightarrow \mathbb{C}^n$ is bounded. (5 marks)
- (b) Show that if $T: \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a bounded linear map, then the spectrum of T is the set of eigenvalues of T . (4 marks)

- (iii) Determine the spectrum of each of the following operators from $L^2[0, 1]$ to $L^2[0, 1]$:

$$R(f)(x) = xf(x) \quad S(f)(x) = (x^2 + 2)f(x)$$

(11 marks)

- 5 (i) State what is meant when we say a subset of a normed vector space is compact. (3 marks)

- (ii) Let V and W be normed vector spaces. State the definition of a compact operator between V and W . (3 marks)

- (iii) Prove that any bounded linear map with finite-dimensional image is compact. You may use without proof the fact that a closed bounded subset of a finite-dimensional normed vector space is compact. (4 marks)

- (iv) Prove that if $T: V \rightarrow V$ is a bounded linear operator, and $K: V \rightarrow V$ is a compact operator, then the composite $TK: V \rightarrow V$ is compact. (5 marks)

- (v) Write down an example of a linear map which is *not* compact. Why not? (5 marks)

- (vi) Prove that the linear map $S: \ell^2 \rightarrow \ell^2$ defined by the formula

$$S(a_1, a_2, a_3, \dots) = \left(a_1, \frac{a_2}{2}, \frac{a_3}{3}, \dots \right)$$

- is compact. You may use without proof the fact that the norm-limit of a sequence of compact linear maps is also compact. (5 marks)

End of Question Paper