The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2014–2015**

Game Theory **2 hours and 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) Members of a group of twenty people are asked to choose independently and simultaneously an integer from 1 to 10. The choices are then tabulated and the payoffs are given as follows:

- if an integer $1 \leq k \leq 10$ was chosen strictly more times than any other, those who chose k receive £1, and others receive nothing,
- if no integer was chosen strictly more often then all others, no payments are made.

(a) Describe this situation as a game in strategic form. ***(4 marks)***

(b) Find all pure-strategy Nash equilibria of this game. ***(5 marks)***

(ii) Firms 1 and 2 produce identical products and they control the production levels q_1 and q_2, respectively. For both companies the cost of producing q units is given by q^2. The prices of the product depend on the production profile (q_1, q_2) of both companies: $P(q_1, q_2) = 6000 - 2(q_1 + q_2)$.

(a) Find each company's production which is the best response to the other company's production. ***(6 marks)***

(b) Find the production profile which is a Nash equilibrium. ***(4 marks)***

(iii) Alice and Bob, who secretly like each other, are going separately on holiday and must choose simultaneously and independently to go to the seaside, to a ski resort or to Disneyland. Their pleasure from the various outcomes is given in the following table where the rows correspond to Alice's choice, the columns correspond to Bob's choice, and each pair of numbers is the utility obtained by Alice and Bob, respectively.

	seaside	ski	Disneyland
seaside	5,4	2,1	5,-8
ski	1,2	4,5	1,-8
Disneyland	-7,4	-7,1	0,-1

Find all pure-strategy and all mixed-strategy Nash equilibria of this game. ***(6 marks)***

2 (i) Consider a finite, two-player, zero-sum game $G = (S, T, u)$.

(a) Describe the sets Δ^R and Δ^C of mixed strategies of both players. ***(2 marks)***

(b) Define the value of the game G. ***(2 marks)***

(c) Define what an optimal strategy of G is. ***(2 marks)***

(d) Let (p^*, q^*) be a mixed-strategy profile of G for which $u(p^*, q^*)$ equals the value V of G. Show that (p^*, q^*) is a Nash equilibrium. ***(5 marks)***

(ii) Consider the following zero-sum game given in tabular form

	A	B	C
I	1	-1	2
II	-2	0	1
III	0	2	-1

and let V be the value of the game.

(a) Does the game have a saddle point? Justify your answer. ***(3 marks)***

(b) Consider the row-player mixed strategy

$$p^* = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}.$$

Find the best responses of the column-player to p^* and deduce a condition on V. ***(4 marks)***

(c) Consider the column-player mixed strategy

$$q^* = \begin{bmatrix} 3/4 \\ 1/4 \\ 0 \end{bmatrix}.$$

Find the best responses of the row-player to q^* and deduce the value of V. ***(4 marks)***

(d) Can any optimal strategy for the row-player mix strategy II with positive probability? Justify your answer. ***(3 marks)***

3 (i) Alice is considering entering a market currently dominated by Bob. If Alice does not enter, she'll make no profit and Bob will earn £5,000,000. If Alice enters the market Bob can fight Alice or accommodate her, and Alice observes his decision. If Bob fights Alice, Alice can

- fight back, resulting in a loss of £2,000,000 for her and a loss of £1,000,000 for Bob, or
- exit the market, resulting in a loss of £3,000,000 for her and a profit of £1,000,000 for Bob.

If Bob does not fight Alice, Alice can

- compete aggressively, resulting in no profit for her and a loss of £3,000,000 for Bob, or
- cooperate with Bob, resulting in a profit of £1,000,000 for her and £2,000,000 for Bob.

(a) Describe this game using a tree, carefully labelling all its components. ***(5 marks)***

(b) Should Alice enter this market? Justify your answer. ***(5 marks)***

(c) Describe the game in strategic form. ***(5 marks)***

(d) Find all pure-strategy Nash equilibria of the game and among these indicate one which is not subgame perfect. ***(5 marks)***

(ii) Consider the game noughts-and-crosses (also known as tic-tac-toe). Show that the person who goes first has a strategy which guarantees at least a draw. ***(5 marks)***

4 (i) Alice (the row-player) and Bob (the column-player) play repeatedly the game G given in tabular form as follows

	A	B
I	3, 3	0, 5
II	5, 0	1, 1

After each repetition of the game, the game is played again with probability p.

(a) What are the expected payoffs for both players if the strategy profile (I, A) is played repeatedly. ***(2 marks)***

(b) If Alice always plays II, what is the largest expected payoff that Bob can get in this repeated game? ***(2 marks)***

(c) Consider the strategy profile for the repeated game in which Alice always plays I and Bob always plays A. Is this a Nash equilibrium? Justify your answer. ***(2 marks)***

(d) Consider the strategy profile for the repeated game in which Alice always plays II and Bob always plays B. Is this a Nash equilibrium? Justify your answer. ***(2 marks)***

(e) Find a $p_0 < 1$ such that for all $p_0 \leq p \leq 1$ one can find a Nash equilibrium with expected payoffs as in (a). Describe this Nash equilibrium in detail. ***(6 marks)***

(ii) Alice and Bob meet at a party and a common friend asks them whether they would like to go on a date. They need to reply to the question simultaneously and independently. Alice heard a lot about Bob and knows him well, whereas Bob is not sure whether this is the clever Alice his friend told him about or the dull Alice his friend mentioned before. He estimates the probability of this being clever Alice at 60%.

The utilities of various responses are given as follows

Alice is clever

	Accept	Decline
Accept	10, 10	-2, 0
Decline	0, -2	0, 0

Alice is dull

	Accept	Decline
Accept	2, -10	-2, 0
Decline	-1, 0	0, 0

(a) If one models this as a Bayesian game, what are the sets of strategies for both players? ***(2 marks)***

(b) Find all Bayes-Nash equilibria of this game. ***(9 marks)***

End of Question Paper