The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2014–2015**

MAS6450 Waves and Magnetohydrodynamics **2 hours**

Answer all four questions.

1 (i) State the induction equation for a perfectly conducting medium. Given a velocity field $\mathbf{v} = (-x, 0, z)$ and the initial condition for the magnetic field

$$\mathbf{B}(\mathbf{x}, 0) = B_0(0, 0, \frac{1}{1+x^2}),$$

find $\mathbf{B}(\mathbf{x}, t)$ by obtaining the Lagrangian coordinates corresponding to $\mathbf{v}$ and applying the Cauchy solution. ***(11 marks)***

Hint: Use an initial condition $\mathbf{x}(0) = (a_1, a_2, a_3)$ and make use of $\mathbf{x}(0)$ for the initial magnetic field $\mathbf{B}(\mathbf{x}, 0) = \mathbf{B}_0$.

(ii) Consider the diffusion of a current sheet in cartesian coordinates with magnetic field $\mathbf{B} = B_z(x, 0)\hat{\mathbf{z}}$ that initially satisfies

$$B_z(x, 0) = \begin{cases} B_0, & x > 0, \\ -B_0, & x < 0. \end{cases}$$

If the magnetic field $\mathbf{B} = (0, 0, B_z)$ is given by

$$B_z(x, t) = \frac{2B_0}{\sqrt{\pi}} \int_0^{x/\sqrt{4\eta t}} e^{-u^2} du,$$

where η is magnetic diffusivity, find the current density and briefly describe the evolution of the current sheet with time near $x = 0$. ***(7 marks)***

Note that, in the above the integral of the Gaussian is $\sqrt{\pi}$.

 Question 1 continued on next page

1 (continued)

(iii) Determine the rate of change of magnetic energy

$$\frac{dW}{dt} = -\int \frac{J^2}{\sigma} dV$$

where J is the current density determined in (ii), V is the volume and $\sigma = (\mu_0 \eta)^{-1}$. Here, the magnetic energy is converted into Joule heating at a rate J^2/σ per unit volume. State clearly, what happens to the magnetic energy over time. ***(7 marks)***

(Hint: Consider length scales in the y and z direction to be l_y and l_z respectively. Note that B_z depends on x only.)

2 (i) State the linearised continuity equation assuming the equilibrium state to be at rest. ***(2 marks)***

(ii) Consider an ideal plasma atmosphere where the equilibrium state is at rest and all perturbed quantities depend on z and t only. Write down the linearised momentum equation with pressure and magnetic force terms. State clearly the equilibrium and perturbed quantities. ***(3 marks)***

(iii) For an isothermal atmosphere, i.e. one satisfying $c_s^2 = \frac{p(z)}{\rho(z)}$ where c_s is the sound speed, $p(z)$ is the pressure and $\rho(z)$ is the density, write the cartesian components of the momentum equation for the equilibrium magnetic field $\mathbf{B_0} = (0, B_0 \sin\theta, B_0 \cos\theta)$. [Hint: take $p_1 = c_s^2 \rho_1$] ***(7 marks)***

(iv) For a perfectly conducting fluid, the perturbed magnetic field, $\mathbf{B}_1$, is governed by

$$\frac{\partial \mathbf{B_1}}{\partial t} = \nabla \times (\mathbf{v_1} \times \mathbf{B_0}) . \qquad (*) \qquad (1)$$

Note that $\nabla \cdot \mathbf{B_1} = 0$. For a perturbed fluid velocity (v_{1x}, v_{1y}, v_{1z}), write down the cartesian components of (*). Use $\mathbf{B_0}$ given in 2(iii). ***(4 marks)***

(v) Assuming perturbed quantities of the form $exp(i\omega t - kz)$ with ω as the frequency and k as the wavenumber, derive the relationship between ω and k using the x-component of the momentum equation and x component of the induction equation (*). If $\theta = 0°$, write down the dispersion relation in terms of the Alfven speed. ***(4 marks)***

(vi) In the Earth's lower stratosphere, the air is isothermal. Use the hydrostatic equilibrium condition to show that

$$p(z) \propto exp(-z/H)$$

where $H = k_B T/(m_p \mu g)$ is the scale height. (Hint: you may use $p = \rho \frac{k_B T}{m_p \mu}$.) ***(5 marks)***

3 (i) Sketch the field lines of the two-dimensional magnetic field whose components are

$$B_x = -2y, \qquad B_y = 4x.$$

State the extreme values of x for the field lines which passes through the point (0,1). ***(10 marks)***

(ii) If a magnetic field $\mathbf{B}$ in a cylindrical geometry (r, ϕ, z) is of the form

$$\mathbf{B} = \frac{B_0}{1 + r^2 a^2} (0, ar, 1),$$

find the z component of the current density.
You may use:

$$\nabla \times \mathbf{G} = \left(\frac{1}{r}\frac{\partial G_z}{\partial \phi} - \frac{\partial G_\phi}{\partial z} \right) \hat{\mathbf{r}} + \left(\frac{\partial G_r}{\partial z} - \frac{\partial G_z}{\partial r} \right) \hat{\phi} + \frac{1}{r}\left(\frac{\partial}{\partial r}(r G_\phi) - \frac{\partial G_r}{\partial \phi} \right) \hat{\mathbf{z}}$$

(5 marks)

(iii) Show that, for the case of non-constant diffusivity, (i.e. $\eta = \dfrac{1}{\mu_0 \sigma}$ is non-constant),

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B} - \nabla \eta \times (\nabla \times \mathbf{B}).$$

(6 marks)

You may use:

Ohm's law $\mathbf{J} = \sigma (\mathbf{E} + \mathbf{v} \times \mathbf{B})$,

Faraday's law $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$,

and the vector identity: $\nabla \eta \times (\nabla \times \mathbf{B}) = \eta \nabla^2 \mathbf{B} - \nabla \times [\eta (\nabla \times \mathbf{B})]$.

(iv) Define the magnetic Reynold's number (R_m) for a constant diffusivity coefficient i.e. $\eta =$ constant, and write down the induction equation when $R_m >> 1$. ***(4 marks)***

4 (i) A magnetic field is given by

$$\mathbf{B} = \frac{B_0}{r^{2p}}(y, -x, 0)$$

where $r^2 = x^2 + y^2$, B_0 is a positive constant and $2p > 1$.

(a) Sketch the field lines. Indicate clearly the direction of the field line in each case. ***(6 marks)***

(b) Discuss the magnetic tension and magnetic pressure gradient forces acting on a parcel of fluid. ***(2 marks)***

(c) Determine the value of p for which the forces in (b) are in balance. ***(4 marks)***

(ii) Let V be a volume enclosed by a surface S. Let B_n be the normal component of the magnetic field $\mathbf{B}$ on the surface S. Show that if B_n is specified then the potential solution $\mathbf{B} = \nabla\phi$ is unique, by considering two potential solution ϕ_1 and ϕ_2 and using the divergence theorem for $\int \nabla\phi^* \cdot \nabla\phi^* \, dV$, where $\phi^* = \phi_1 - \phi_2$. ***(6 marks)***

(Hint: You may use $\nabla\phi^* \cdot \nabla\phi^* = \nabla \cdot (\phi^*\nabla\phi^*) - \phi^*\nabla^2\phi^*$)

(iii) Assume a cylindrical coordinates (R, ϕ, z). If a magnetic field $\mathbf{B} = \mathbf{B}(R)$ varies with R alone, why can it not possess a radial component B_R? ***(3 marks)***

(iv) Use the equations

$$\rho_0 \frac{\partial \mathbf{v_1}}{\partial t} = -\nabla p_1,$$

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{v_1} = 0,$$

$$p_1 = c_s^2 \rho_1,$$

to show that a wave corresponding to perturbations of the form $\rho_1 \propto e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ has frequency $\omega^2 = c_s^2 k^2$.
Use the above equations to show that $\mathbf{v_1}$ is parallel to the direction of propagation. ***(4 marks)***

End of Question Paper