The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2014–2015**

Stochastic Processes and Finance **3 hours**

Candidates may bring to the examination a calculator that conforms to University regulations. Full marks may be obtained by complete answers to five questions. All answers will be marked, but credit will be given only for the five best answers.

1 Let $\Omega = \{1, 2, 3, 4, 5\}$ with $\mathcal{F}$ being the power set of Ω (the collection of all subsets of Ω). Let $\mathbb{P}$ be the uniform probability measure on Ω, that is $\mathbb{P}(\{i\}) = 1/5$ for all $1 \leq i \leq 5$. Let

$$A = \{1, 2, 3\}, \quad B = \{3, 4, 5\}.$$

(a) Show that for two subsets C and D of *any (not just the one considered in this problem)* space Ω we have $\mathbf{1}_C \cdot \mathbf{1}_D = \mathbf{1}_{C\cap D}$. Note that for a set C, $\mathbf{1}_C$ denotes the indicator function of C:

$$\mathbf{1}_C(i) = \begin{cases} 1 & \text{if } i \in C \\ 0 & \text{if } i \notin C \end{cases}$$

(3 marks)

(b) Recall that $\mathbb{E}(\mathbf{1}_C) = \mathbb{P}(C)$. For the sets A and B defined above, compute $\mathbb{E}\left[\left(2\mathbf{1}_A + 3\mathbf{1}_B\right)^2\right]$. ***(4 marks)***

(c) Are the sets A and B above independent? ***(2 marks)***

(d) Consider the probability measure $\mathbb{Q}$ on Ω given by $\mathbb{Q}(\{1\}) = 1/4$, $\mathbb{Q}(\{2\}) = 3/4$ and $\mathbb{Q}(\{i\}) = 0$ for $i \neq 1, 2$. Is $\mathbb{Q}$ absolutely continuous with respect to $\mathbb{P}$? Are both $\mathbb{Q}$ and $\mathbb{P}$ equivalent? Explain. ***(4 marks)***

(e) (i) For an integrable random variable X and a sub σ-algebra $\mathcal{G}$ of $\mathcal{F}$, write down the definition of the conditional expectation $\mathbb{E}(X|\mathcal{G})$. ***(3 marks)***

(ii) Let A and B be as above. Consider the σ-algebra $\mathcal{G} = \{\emptyset, B, B^c, \Omega\}$. Using your definition in (i), show that the conditional probability of A given $\mathcal{G}$ is

$$\mathbb{P}\big(A|\mathcal{G}\big)(i) := \mathbb{E}(\mathbf{1}_A|\mathcal{G})(i) = \begin{cases} \frac{1}{3}, & i \in B \\ 1, & i \in B^c \end{cases}$$

You might use the following fact without proving it. Any $\mathcal{G}$ measurable random variable can be written as $c\mathbf{1}_B + d\mathbf{1}_{B^c}$ where c and d are constants. ***(4 marks)***

Hint: Find c and d.

2 Let S_n denote the size of a population of bacteria at time n with $S_0 = 1$. The evolution of the population is as follows. At time n, each bacteria present *independently* splits into *two* with probability $1/2$ or *dies* with probability $1/2$. Thus the number of bacteria present at time $n+1$ is given by

$$S_{n+1} = \begin{cases} X^{(n)}(1) + X^{(n)}(2) + \cdots + X^{(n)}(S_n) & \text{if } S_n > 0 \\ 0 & \text{otherwise,} \end{cases}$$

where the $\{X^{(n)}(i),\, i \geq 1,\, n \geq 0\}$ are i.i.d. random variables which take the values 0 or 2 with probability $1/2$ each.

(a) Show that $S_{n+1} = \sum_{i=1}^{\infty} \mathbf{1}_{\{i \leq S_n\}} X^{(n)}(i)$. ***(2 marks)***

(b) Show that $S_n = \sum_{i=1}^{\infty} \mathbf{1}_{\{i \leq S_n\}}$. ***(2 marks)***

(c) Explain why $S_{n+1} \leq 2S_n$. ***(2 marks)***

(d) Write down the definition of a martingale. ***(3 marks)***

(e) Show that S_n is a martingale with respect to the filtration $\mathcal{F}_n = \sigma\{X^{(k)}(i),\, i \geq 1,\, k \leq n-1\}$. ***(5 marks)***

(f) Find $\mathbb{E}\left(\exp\left\{tX^{(n)}(i)\right\}\right)$. ***(2 marks)***

(g) Compute $\mathbb{E}\big[\exp(tS_2)\big]$. *Hint: Condition on S_1.* ***(4 marks)***

3 Consider a *Brownian motion* $\{B(t),\, t \geq 0\}$.

(a) Define what is meant by a Brownian motion. ***(3 marks)***

(b) Show that $2B(t/4)$, $t \geq 0$ is also a Brownian motion. ***(3 marks)***

(c) Show that $B(t)^3 - 3tB(t)$ is a martingale with respect to the natural filtration. ***(6 marks)***

Fact: For $X \sim N(0, \sigma^2)$ we have that $\mathbb{E}[X^3] = 0$.

(d) Find the SDE satisfied by $X(t) = \exp\{t + B(t)^2\}$ and write the solution in integral form. *Hint: Itô's formula.* ***(4 marks)***

(e) Find the SDE satisfied by $Y(t) = B(t)X(t)$. ***(4 marks)***

4 Consider a finite market defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathcal{F}_n$, $n = 0, 1, \cdots T$ satisfying $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_T = \mathcal{F}$. There are $d+1$ assets with prices given by the *adpated* process $S(n) = \big(S_0(n),\, S_1(n), \cdots, S_d(n)\big)$ where $n = 0, 1, \cdots T$. Here S_0 is the price of a risk-free asset and $S_1, S_2, \cdots, S_d$ are the prices of stocks. Let $\phi(n) = \big(\phi_0(n),\, \phi_1(n), \cdots, \phi_d(n)\big)$ be a trading portfolio.

(a) A *self-financing* portfolio satisfies $\phi(n) \cdot S(n) = \phi(n+1) \cdot S(n)$. Explain in words what this means. ***(3 marks)***

(b) Suppose ϕ is a self-financing portfolio. Prove that

$$V_\phi(n) - V_\phi(0) = \sum_{m=1}^{n} \phi(m) \cdot \big[S(m) - S(m-1)\big]$$

where $V_\phi(n) = \phi(n) \cdot S(n)$. ***(4 marks)***

(c) Suppose X is an attainable European contingent claim and ϕ and ξ are two distinct replicating strategies (portfolios). Show that an arbitrage opportunity exists if $V_\phi(0) \neq V_\xi(0)$. ***(5 marks)***

(d) Define a martingale measure $\mathbb{P}^*$ for the discounted asset price process $\tilde{S}(n) = S_0(n)^{-1} S(n)$. ***(2 marks)***

(e) Note that a portfolio is *previsible* in the sense that each $\phi_i(n)$ is $\mathcal{F}_{n-1}$ measurable. If ϕ is a self-financing portfolio and $\mathbb{P}^*$ is a martingale measure, deduce that the discounted wealth process $\widetilde{V}_\phi(n) = \phi(n) \cdot \tilde{S}(n)$, $n = 0, 1, \cdots, T$ is a $\mathbb{P}^*$ martingale. ***(6 marks)***

5 (a) Consider the Binomial Asset Pricing model with a bond and a stock. The price of the bond at time n is $B(n) = (1+r)^n$ and the stock price is given by $S(n) = S(0)\big(1 + K(1)\big)\big(1 + K(2)\big) \cdots \big(1 + K(n)\big)$ where the $K(n)$ are i.i.d taking values u and d with probability p and $1-p$ respectively. Here $u > d > -1$ and $0 < p < 1$. *Assume* $d < r < u$ so that the market is arbitrage-free and complete.

(i) Give the system of equations that the price process $\pi_A(n)$ of an American contingent claim with pay-off $X(n)$ must satisfy. What is the interpretation of these equations? ***(5 marks)***

(ii) The payoff for a call option with exercise price k is given by $X(n) = \max(S(n) - k, 0)$. Show that price process $\pi_A(n)$ of an American call option is the same as the price process $\pi_E(n)$ of a European call option.
Hint: Show that $\pi_E(n) \geq X(n)$. ***(10 marks)***

(b) In the context of the term structure of bonds, state the Martingale Hypothesis. Make sure you define all the quantities used in the hypothesis. ***(5 marks)***

6 Consider the Black-Scholes model with two assets: a bond and a stock, during the time $0 \le t \le 5$. The price of the bond at time t is given by $A(t) = e^t$ and the price $S(t)$ of the stock at time t satisfies the SDE: $dS(t) = 2S(t)\,dt + S(t)\,dB(t)$ with $S(0) = 1$. Suppose $\phi(t) = 1$ for all t where $\phi(t)$ is the number of stocks at time t.

(a) What can you say about $\psi(t)$ (the number of bonds at time t) if the portfolio is self-financing? Assume that ψ is an Itô process. ***(6 marks)***

(b) The measure $\mathbb{Q}$ under which the discounted stock price $\tilde{S}(t)$ is a martingale is given by

$$d\mathbb{Q} = \exp\left\{-B(5) - \frac{5}{2}\right\} \cdot d\mathbb{P}.$$

(i) We know that under the measure $\mathbb{Q}$, $d\tilde{S}(t) = \tilde{S}(t)\,dW(t)$ where $W(t)$ is a Brownian motion. Show that $\tilde{S}(t) = \exp\left\{W(t) - \frac{t}{2}\right\}$ is a solution to the equation. ***(4 marks)***

(ii) Compute the price of the European contingent claim $X = S(5)^2$ at time 0. How many bonds should we have at time 0 to attain X at time 5? ***(10 marks)***

Hint (completing the squares): You might need $x^2+2ax = (x+a)^2-a^2$.

Formula: The density of the Normal distribution $N(\mu, \sigma^2)$ with mean μ and variance σ^2 is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{\frac{-(x-\mu)^2}{2\sigma^2}\right\}, \quad x \in \mathbb{R}.$$

End of Question Paper