The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring 2014-2015**

Financial Mathematics 2 **hours and** 30 **minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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1 Consider the following three bonds with face value of £100:

Time to maturity (in years)	Annual interest (paid every 6 months)	Bond price (in £)
0.5	4%	101.49
1.	8%	105.92
1.5	6%	105.13

(i) Find the 0.5-year spot interest rate. ***(1 mark)***

(ii) Use the bootstrap method to find the 1 and 1.5-year spot interest rates. ***(8 marks)***

(iii) Consider a forward contract to deliver the 1.5-year bond listed in the table above, with delivery date in 1 year. (Assume that the bond is delivered immediately after the payment of its second coupon.) What is the correct forward price for this contract? ***(3 marks)***

(iv) Suppose you are offered a short position in the forward contract described in (iii) with a forward price of £104. Exhibit in detail an arbitrage strategy available to you. ***(13 marks)***

2 (i) (a) Consider a portfolio consisting of European put options on the same underlying asset and same expiration date, which is short two options with strike 30, short one option with strike 50 and long one option with strike 70. Sketch a graph of the payoff function of this portfolio. ***(4 marks)***

(b) Let p_{30}, p_{50} and p_{70} be the spot prices of the put options in part (a) with strike prices 30, 50 and 70, respectively. Let c_{10} be the spot price of a European call option on the same underlying asset and with the same expiration date as the options above and with strike price 10. Describe an inequality involving c_{10}, p_{30}, p_{50} and p_{70} and explain in detail why it holds. ***(7 marks)***

(ii) The price of a stock which pays no dividends is currently £15 and at the end of one year its price will be either £20 or £10. Suppose that all interest rates are constant and equal to 3%. Consider an American put option on this stock expiring in one year and with an unspecified strike price of £X, where $10 \leq X \leq 20$. This put option can be exercised either at expiration or immediately.

(a) Describe a portfolio consisting of some number of units of the stock described above together with one of the American put options described above with the property that, if the put option is held until expiration, the value of this portfolio in one year is the same regardless of the stock price. (Your answer will involve the unknown value of X.) ***(7 marks)***

(b) Find all values of $10 \leq X \leq 20$ for which the put option above should be exercised immediately. ***(7 marks)***

3 (i) (a) State the mathematical definition of Brownian motion. ***(5 marks)***

(b) State Ito's Lemma. ***(3 marks)***

(ii) Assume that a stock price S is given as the Ito process

$$dS = \mu S\, dt + \sigma S\, dB$$

where μ and σ are constants. Let $f = f(S,t)$ be the value at time t of a derivative contingent on the value of S at some time T. Assume further that $f(s,t)$ is twice continuously differentiable with respect to s and continuously differentiable with respect to t. Assume also that all interest rates are non-stochastic and equal to r.

(a) Show that the process followed by $f(S,t)$ is

$$df = \left(\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial f}{\partial S} \sigma S dB.$$ ***(3 marks)***

(b) Consider a portfolio consisting of a variable quantity $\dfrac{\partial f}{\partial S}(S,t)$ of shares and -1 derivatives; let Π be the value of this portfolio, i.e., $\Pi = \dfrac{\partial f}{\partial S}(S,t)S - f$. Show that after a short period of time Δt the value of the portfolio changes by

$$\Delta \Pi \approx \left(-\frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 - \frac{\partial f}{\partial t} \right) \Delta t$$

(5 marks)

(c) Deduce that $\Delta \Pi \approx r \Pi \Delta t$. ***(4 marks)***

(d) Deduce that

$$\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf.$$ ***(2 marks)***

(e) How does the risk-aversion of investors affect the price of this derivative? Justify your answer in detail. ***(3 marks)***

4 (i) Define the following concepts in the context of Portfolio Theory.

(a) The *market portfolio*. ***(2 marks)***

(b) The *capital market line*. ***(2 marks)***

(c) The *beta-coefficient* of an investment. ***(2 marks)***

(ii) Sketch the following:

(a) an example of a feasible set and efficient frontier in the absence of a risk-free investment, ***(2 marks)***

(b) an example of a feasible set, market portfolio and efficient frontier in a market containing a risk-free investment. ***(3 marks)***

(iii) You are given the following data on three stocks and the market portfolio:

	Expected return	Correlation with market portfolio	Standard deviation of return
Stock 1	?	0.7	80%
Stock 2	5%	?	63%
Stock 3	3.8%	0.9	?
Market portfolio	4.5%	1	37%

The risk-free interest rate for the period is 3%. Give the equation of the capital market line, find the beta-coefficients of Stocks 1, 2 and 3, and fill in all missing data in the table above. ***(14 marks)***

End of Question Paper