The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2014–2015

MAS6340: Analysis I — **2 hours 30 minutes**

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

Throughout this paper, unless otherwise stated, all normed vector spaces are either over the field of real numbers, $\mathbb{R}$, or the field of complex numbers, $\mathbb{C}$.

1 (i) State what is meant by the statement that a normed vector space is a *Banach space.* ***(3 marks)***

(ii) Let V be the space $\mathbb{R}^k$ with the norm

$$\|(\alpha_1, \dots, \alpha_k)\| = \left(\alpha_1^4 + \cdots + \alpha_k^4\right)^{\frac{1}{4}}.$$

Show that V is a Banach space. ***(6 marks)***

(iii) Define what is meant by a *bounded linear operator* $T\colon V \to W$ between normed vector spaces V and W. ***(2 marks)***

(iv) Show that a linear map $T\colon V \to W$, where V and W are normed vector spaces, is continuous if and only if it is a bounded linear operator. ***(9 marks)***

(v) Let V be the vector space $\mathbb{R}^k$ with the norm

$$\|(\alpha_1, \dots, \alpha_k)\| = |\alpha_1| + \cdots + |\alpha_k|.$$

Show that every linear map $T\colon V \to W$ is continuous. ***(5 marks)***

2 Let H be a complex Hilbert space. Throughout this question, you may use without proof the fact that if V is a closed linear subspace of H, then $H = V \oplus V^{\perp}$. You may also use without proof the Cauchy-Schwarz inequality, and the fact that the inner product is continuous.

(i) For a linear subspace $U \subseteq H$, prove that $\overline{U}^{\perp} = U^{\perp}$, and $U^{\perp}$ is closed. ***(6 marks)***

(ii) Let $f\colon H \to \mathbb{C}$ be a continuous linear map. Prove that we have a unique element $R_f \in H$ such that

$$f(x) = \langle R_f, x \rangle$$

for all $x \in H$. ***(9 marks)***

(iii) Let l^2 be the Hilbert space

$$\left\{ (a_n) \ \mid \ \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\}$$

with the usual inner product. Define $f, g\colon l^2 \to \mathbb{C}$ by

$$f((a_n)) = a_1 \qquad g((a_n)) = \sum_{n=4}^{\infty} \frac{a_n}{n}.$$

Find R_f, R_g, and R_{f+ig}. ***(5 marks)***

(iv) Let $T\colon H \to H$ be a bounded linear map, and let $U \subseteq H$ be a closed linear subspace. Is it necessarily true that $T[U^{\perp}] = T[U]^{\perp}$? Justify your answer. ***(5 marks)***

3 (i) State Zorn's lemma. ***(4 marks)***

(ii) Define what is meant by an *orthonormal basis* for a Hilbert space H. Prove that every Hilbert space has an orthonormal basis. ***(11 marks)***

(iii) Define the *graph* of a linear map $T\colon V \to W$. State the closed graph theorem. ***(4 marks)***

(iv) Let H be a Hilbert space, and let $f, g\colon H \to H$ be linear maps such that

$$\langle f(x), y \rangle = \langle x, g(y) \rangle$$

for all $x, y \in H$. Prove that f and g are both continuous. ***(6 marks)***

4 (i) Let A be a unital Banach algebra, and let $x \in A$. Prove tht if $\|x\| < 1$, then the element $1 - x$ is invertible. ***(8 marks)***

(ii) Define the *spectrum* of an element $x \in A$, and the *spectral radius* $R_\sigma(x)$. ***(3 marks)***

(iii) Prove that for any $x \in A$, $R_\sigma(x) \leq \|x\|$. ***(4 marks)***

(iv) Let H be a Hilbert space. Say what is meant by the statement that a bounded linear map $U \colon H \to H$ is a *unitary operator*. ***(2 marks)***

(v) Prove that if $U \colon H \to H$ is a unitary operator, then $\|U\| = 1$. ***(4 marks)***

(vi) Prove that if $U \colon H \to H$ is unitary, then

$$\mathrm{Spectrum}(U) \subseteq \{z \in \mathbb{C} \mid |z| = 1\}.$$

(4 marks)

5 (i) Let V be a normed vector space. Define what is meant by a *compact operator* $K \colon V \to V$. ***(3 marks)***

(ii) Prove that any bounded linear map with finite-dimensional image is a compact operator. You may use the Heine-Borel theorem without proof. ***(4 marks)***

(iii) Prove that any compact operator is a bounded linear map. ***(4 marks)***

(iv) Give an example of a bounded linear map that is *not* a compact operator. Justify your answer. ***(4 marks)***

(v) Let H be a Hilbert space. Define what is meant by a *Fredholm operator* $T \colon H \to H$ and its *Fredholm index*. ***(3 marks)***

(vi) Prove that each of the following operators on the Hilbert space l^2 are Fredholm, and calculate their Fredholm indices. In the calculation, you may use any standard properties of the Fredholm index you need.

(a) $R(a_1, a_2, a_3, a_4, a_5, \ldots) = (0, 0, a_1, a_2, a_3, a_4, a_5, \ldots)$

(b) $S(a_1, a_2, a_3, a_4, a_5, a_6, a_7, \ldots) = (a_1, a_2, a_1, a_2, a_3, a_4, a_5, a_6, a_7, \ldots)$

(c) $T(a_1, a_2, a_3, a_4, a_5, \ldots) = (0, 0, 0, a_2, a_3, a_4, a_5, \ldots)$

(7 marks)

End of Question Paper