The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2014–2015**

Algebraic Topology I **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 In this question you need to justify all your answers. If you claim that a statement is true you need to prove it and if you claim it is false you need to give a counterexample. You may quote results from the lectures as part of your justifications.

(i) Let $f, g : I \longrightarrow S^1$ be two maps from the unit interval $I = [0, 1]$ to the unit sphere S^1 in $\mathbb{R}^2$ defined as follows:

$$f(x) = (\cos(6\pi x), \sin(6\pi x))$$

$$g(x) = (1, 0)$$

(a) Is f homotopic to g? ***(2 marks)***

(b) Is f loop homotopic to g? ***(2 marks)***

(ii) Is the two-dimensional real projective space $\mathbb{R}P^2$ homotopy equivalent to the two-dimensional sphere S^2? ***(3 marks)***

(iii) (a) Let D^2 be the unit disc in $\mathbb{R}^2$,

$$D^2 = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leq 1\}.$$

Is D^2 contractible? If so give a direct proof, if not give a reason. ***(3 marks)***

(b) Is the complement of the disc in the plane, $\mathbb{R}^2 \setminus D^2$, contractible? ***(3 marks)***

(iv) Calculate $\pi_1(\mathbb{R}P^2 \times T)$, where $\mathbb{R}P^2$ is two-dimensional real projective space and T is the torus. ***(3 marks)***

(v) When (if ever) can the fundamental group of a space depend on the choice of a basepoint? Illustrate your answer with examples and any appropriate results from lectures. ***(4 marks)***

2 As previously we use the notation S^1 for the one dimensional sphere, D^2 for the two-dimensional disc and $\mathbb{R}P^2$ for two-dimensional real projective space. $S^1 \vee D^2$ denotes a wedge sum of S^1 and D^2 (where the base point for D^2 is chosen on the boundary) and $S^1 \vee \mathbb{R}P^2$ denotes a wedge sum of S^1 and $\mathbb{R}P^2$.

(i) (a) What is $\pi_1(S^1 \vee D^2)$? ***(1 mark)***

(b) Calculate $\pi_1(S^1 \vee \mathbb{R}P^2)$ using Van Kampen's Theorem. ***(4 marks)***

(c) What is the universal cover for each of the spaces S^1, D^2 and $S^1 \vee D^2$? You are encouraged to draw pictures as part of your answer. ***(3 marks)***

(d) Characterise all connected covers of $S^1 \vee D^2$ and prove that what you listed is the full characterisation. You are encouraged to draw pictures as part of your answer. ***(6 marks)***

(ii) Construct a space X with $\pi_1(X) = \langle a, b, c \mid a^2bc, acb \rangle$ and prove that $\pi_1(X)$ is as requested. ***(6 marks)***

3 (i) Let X and Y be topological spaces.

(a) Define the *singular chain complex* $C_*(X)$ of X. ***(3 marks)***

(b) Given a continuous function $f : X \to Y$, explain how to define the group homomorphism $C_n(f) : C_n(X) \to C_n(Y)$, for $n \geq 0$. ***(2 marks)***

(c) Show that the sequence of homomorphisms $\{C_n f\}$ gives a chain map $C_*(f) : C_*(X) \to C_*(Y)$. ***(3 marks)***

(ii) Calculate all the homology groups of the chain complex C:

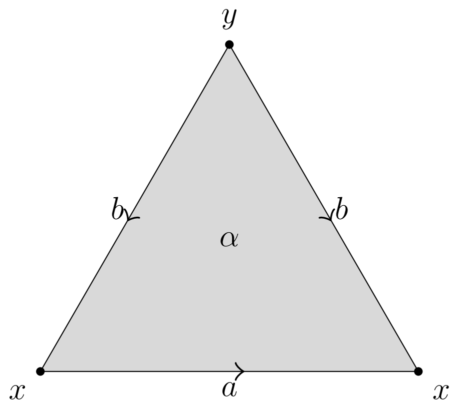
$$0 \xrightarrow{\delta_4} \mathbb{Z}\{a\} \xrightarrow{\delta_3} \mathbb{Z}\{b\} \oplus \mathbb{Z}\{c\} \oplus \mathbb{Z}\{d\} \xrightarrow{\delta_2} \mathbb{Z}\{e\} \oplus \mathbb{Z}\{f\} \xrightarrow{\delta_1} \mathbb{Z}\{g\} \xrightarrow{\delta_0} 0$$

where $C_n = 0$ for $n \geq 4$ and

$$\delta_3(a) = 7d - 7b,$$
$$\delta_2(b) = 15e - 6f, \quad \delta_2(c) = 30e - 12f, \quad \delta_2(d) = 15e - 6f,$$
$$\delta_1(e) = 2g, \quad \delta_1(f) = 5g.$$

(The notation $\mathbb{Z}\{x\}$ means the free abelian group on generator x.) ***(7 marks)***

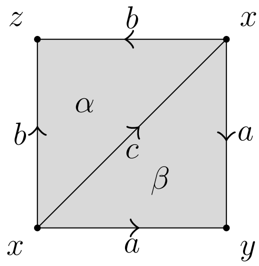
(iii) Consider the Δ-complex shown in the diagram below, a 2-simplex with identifications as indicated.



(a) Write down the chain complex associated to this Δ-complex and calculate all its homology groups. ***(4 marks)***

(b) Explain the answer obtained in part (a) in geometric language. ***(1 mark)***

4 (i) Consider the cell complex X as pictured below.



Let A be the subcomplex consisting of the 0-cell x and the 1-cell c.

(a) Write down the chain complexes of X and A and the relative chain complex $C_*(X, A)$. ***(4 marks)***

(b) Calculate the relative homology groups $H_n(X, A)$, for $n \geq 0$. ***(3 marks)***

(c) Describe the quotient space X/A and check that the reduced homology groups of X/A are the same as the relative homology groups calculated in part (b). ***(3 marks)***

(ii) (a) Consider a topological space $X = A \cup B$, where A and B are open subsets of X. Suppose that $\widetilde{H}_n(A) = \widetilde{H}_n(B) = \widetilde{H}_n(A \cap B) = 0$ for all $n \geq 0$. If $A \cap B$ is non-empty, show that $\widetilde{H}_n(X) = 0$ for all $n \geq 0$. ***(4 marks)***

(b) Now consider a topological space $Y = A \cup B \cup C$, where A, B and C are open subsets of Y. Suppose that the reduced homology groups $\widetilde{H}_n$ are zero for all $n \geq 0$ for all of the following spaces: A, B, C, $A \cap B$, $A \cap C$, $B \cap C$, $A \cap B \cap C$. Suppose also that each of the pairwise intersections, $A \cap B$, $A \cap C$ and $B \cap C$, is non-empty.

Show that $\widetilde{H}_n(Y) = 0$ for all $n \geq 2$.

Give an example of this situation where $\widetilde{H}_1(Y) \neq 0$. ***(6 marks)***

End of Question Paper