Data provided: Formulae sheet

The University Of Sheffield.

CIV340

SCHOOL OF MATHEMATICS AND STATISTICS

Autumn Semester 2014-2015

Computational Engineering Mathematics

Three hours

Marks will be awarded for your best FOUR answers

1 (i) Second order linear PDEs of the general form

$$A + B\Phi + C\frac{\partial \Phi}{\partial x} + D\frac{\partial \Phi}{\partial y} + E\frac{\partial^2 \Phi}{\partial x^2} + F\frac{\partial^2 \Phi}{\partial y^2} + G\frac{\partial^2 \Phi}{\partial x \partial y} = 0$$

have arbitrary coefficients A, B, C, D, E, F and G that are constant or depend only on the independent variables.

(a) What condition in respect of these coefficients determines classification of the solution as elliptic, typical of a steady state? ***(1 mark)***

(b) What class of solution is typical with the discriminant $\Delta > 0$? ***(1 mark)***

Give a simple example of a physical system that is modelled by this type of solution. ***(1 mark)***

(c) Identify respectively all regions of the x, y field where the equation

$$y\frac{\partial^2 U}{\partial x^2} + (1 - x)\frac{\partial^2 U}{\partial y^2} = 1$$

is elliptic, parabolic or hyperbolic. ***(3 marks)***

(ii) We seek a solution over the domain $0 \leq x \leq 1$ of the equation

$$\frac{\partial U}{\partial t} = \alpha\frac{\partial^2 U}{\partial x^2} - \beta\frac{\partial U}{\partial x},$$

where α and β are positive real constants.

(a) What is the classification of the solution to this equation? Give a simple example of a physical system that is modelled by this type of solution. ***(1 mark)***

(b) What auxilliary conditions must be specified to obtain the solution? ***(2 marks)***

(c) Let $\alpha = 1$ and $\beta = 0.5$. Use the standard notation $U_{i,j} \equiv U(x_i,\ t_j)$. Construct an explicit finite difference formulation of the equation for $U_{i,j+1}$ accurate in space to $O(h^2)$, where h is the grid separation. ***(3 marks)***

(d) Given the boundary condition at $x = 1.0$ is given by

$$\frac{\partial U}{\partial x} = 0,$$

find the expression for $U_{n,j+1}$, where $x_n = 1.0$ ***(3 marks)***

 Question 1 continued on next page

1 (continued)

(iii) The solution to this equation in implicit form is

$$\left(1+\frac{2\alpha\Delta t}{(\Delta x)^2}\right)U_{i,j+1}+\left(\frac{\beta\Delta t}{2\Delta x}-\frac{\alpha\Delta t}{(\Delta x)^2}\right)U_{i+1,j+1}-\left(\frac{\beta\Delta t}{2\Delta x}+\frac{\alpha\Delta t}{(\Delta x)^2}\right)U_{i-1,j+1}=U_{i,j}.$$

Below is a *Scilab* script to solve the equation using this implicit scheme, with the boundary conditions being $U_{0,j} = 40.0$ and $U_{j,0} = 40.0 - 25.0x^2$. There are four errors of **ordering or omission** (not syntax errors), which would need to be fixed to solve the equation.
Write suitable corrections and state where each would apply, by reference to the line numbers listed in the code. ***(4 marks)***

Explain the physical significance of **Eps** in this code. ***(1 mark)***

Implicit solution

```
00  N = 20;  dx = 1/N; // Note that the number of internal points is N, with U(N+1)
    derived from derivative boundary condition
01  dt = 0.05;  x=linspace(dx,1,N);  maxt = 10;
–   // set up parameters
02  t = 0.0;  alpha = 1.0;  beta = 0.5;  Eps = 100.0;
–   // set up (N − 1) × (N − 1) coefficient matrix (note the tri-diagonal form)
–   // Set up lower-diagonal part :
05  ld = ones(N-1,1);
10  Lower = -(0.5*betas+alpha/dx)*dt/dx*diag(ld,-1);
–   // Set up main-diagonal part:
15  md = ones(N,1);
20  Diag = (1+2*alpha*dt/dx/dx)*diag(md,0);
–   // Set up upper-diagonal part:
25  ud = ones(N-1,1);
30  Upper = (0.5*beta-alpha/dx)*dt/dx*diag(ud,+1);
–   // Now put all together:
35  A = Lower+Diag+Upper;
40  Uold = U;
–   // advance solution using implicit scheme
45  while (Eps > 0.001 ) then;
50  if t > maxt then break;
55  end
60  t = t + dt;
65  b = Uold;
70  b(N) = (1 + 2*alpha*dt/dx/dx)*U(N) + 2*(0.5*beta-alpha/dx)*dt/dx*U(N-1)
75  U = A\b;
80  plot(x,U)
–   // compute the required condition :
85  dUt = (U-Uold)/dt;
90  Eps = norm(dUt,2);
95  end
99  t // Print out how long till profile settles down
```

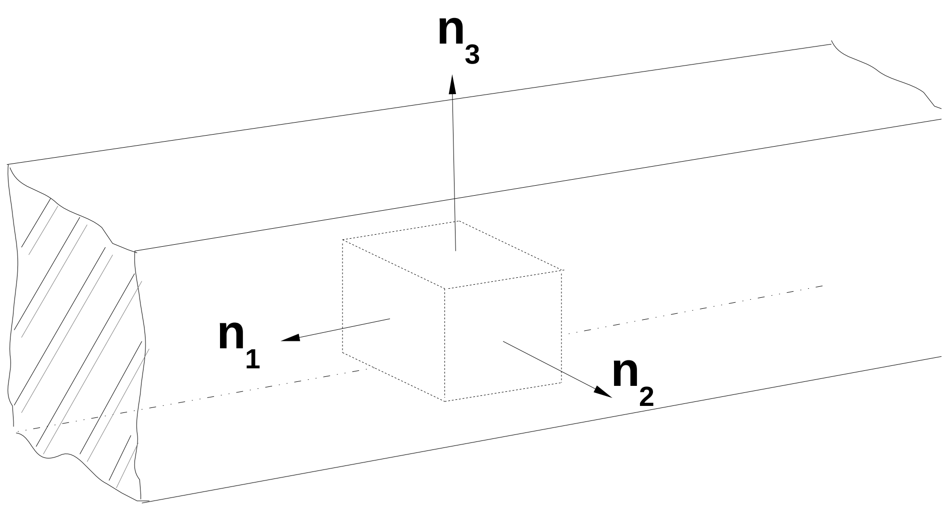


Figure 1: Supporting beam

2 A horizontal support beam is at rest, as shown in Figure 1. An infinitessimally small interior volume segment inside the beam has surface normals $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$. These align to the coordinate axes, respectively, parallel ($\hat{\mathbf{x}}_1$), horizontally perpendicular ($\hat{\mathbf{x}}_2$) and vertically perpendicular ($\hat{\mathbf{x}}_3$) to the beam. There are no horizontal body forces nor horizontal shear stresses σ_{12}.

(i) (a) Derive the general equation of force balance in the vertical direction, by considering surface stresses and the body forces acting within this beam on the volume segment of size $\Delta x_1\, \Delta x_2\, \Delta x_3$. ***(4 marks)***

(b) What are the equations of force balance acting on this beam in the horizontal directions? Which of these are non-zero terms? ***(1 mark)***

(ii) The stess tensor applying within the beam is given by

$$[\sigma] = \begin{bmatrix} x_1 & 0.0 & -x_3 \\ 0.0 & 5x_1 & -2x_1 \\ -x_3 & -2x_1 & 3x_3 \end{bmatrix}$$

and the body force by $\mathbf{F} = (0, 0, \rho g)^T$, where the density of the beam ρ is constant and $g = -10.0\text{m s}^{-2}$.

(a) The stress force per unit area is $\mathbf{f} = [\sigma]\hat{\mathbf{n}}$. Find the total stress forces acting on a rectangular surface in the positive x_1-plane defined by $x_1 = 1.0$, $0.0 \leq x_2 \leq 1.0$ and $0.0 \leq x_3 \leq 0.5$. ***(3 marks)***

(b) From the x_3-component of the equation of force balance find the beam density ρ. ***(2 marks)***

 Question 2 continued on next page

2 (continued)

(iii) A load is applied to the beam causing a small elastic displacement. The engineering strains at a location within the beam are such that

$$\epsilon_{33} = 0.1x_1 + 0.05x_2 + x_3, \quad \gamma_{13} = -0.4x_3, \quad \gamma_{23} = -0.2x_3$$

and otherwise the strains are zero.

(a) The constitutive matrix is $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. Show that $\sigma_{ij} = \lambda\delta_{ij}\epsilon_{mm} + 2\mu\epsilon_{ij}$. ***(3 marks)***

(b) If $\lambda = 2$ and $\mu = 0.5$ obtain the new stress state for this location within the beam σ_{ij}. ***(2 marks)***

(c) Subject to the strain the beam is in a new state of equilibrium. Verify that all three balance of forces equations are satisfied for this new state of stress. ***(2 marks)***

(iv) We model numerically the stress-strain relations for various loads on this beam. What would be the advantages and disadvantages of applying a finite difference approach compared to finite element approach?

(3 marks)

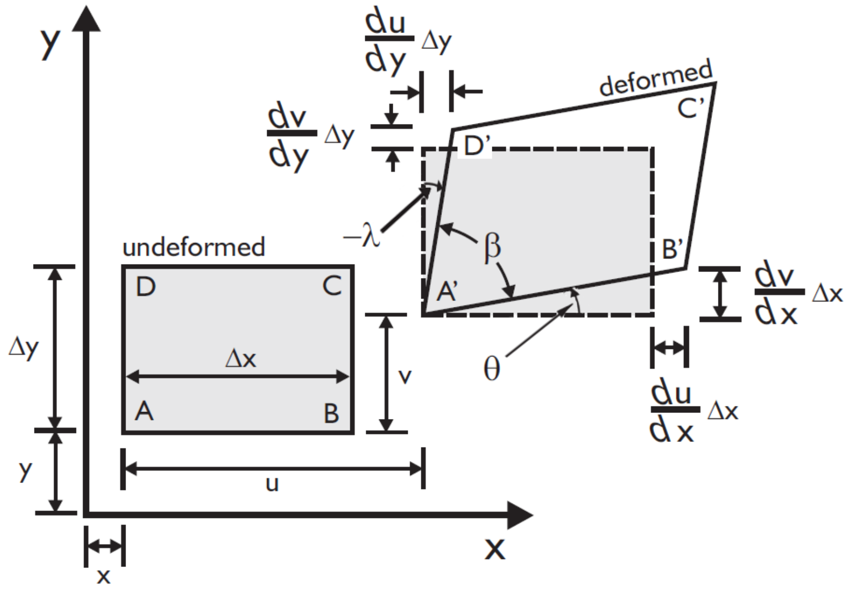


Figure 2: Two-dimensional strain

3 (i) (a) Define the normal strains, ϵ_{xx} and ϵ_{yy}, and the engineering shear strain, γ_{xy}. Infer then the definitions for the remaining strains ϵ_{zz}, γ_{xz} and γ_{yz}. *(2 marks)*

(b) By reference to Figure 2, show that

$$\epsilon_{yy} = \frac{\partial v}{\partial y}, \quad \gamma_{yx} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y},$$

where u is the displacement in the x-direction and v is the displacement in the y-direction of the body ABCD due to the stress forces acting on its surfaces. *(8 marks)*

(ii) The elastic stiffness and the elastic bulk modulus are $E = 27.0$GPa and $K = 110.0$GPa, respectively, for a given homogeneous elastic material. Its elastic constitutive matrix applying to the *tensorial* strains, ϵ_{kl}, on this material can be denoted by

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

(a) With the aid of the table in the formula sheet, either enter the values of $[C]$ appropriately in a (6×6) matrix, or list each non-zero tensor element and its value. *(8 marks)*

(b) A state of strain at some point within this material is defined by $\epsilon_{xx} = 1000 \times 10^{-6}$, $\epsilon_{yy} = -0.30\epsilon_{xx}$, $\epsilon_{zz} = -0.20\epsilon_{xx}$, $\gamma_{xy} = 240 \times 10^{-6}$, $\gamma_{yz} = 400 \times 10^{-6}$ and $\gamma_{zx} = -75 \times 10^{-6}$. Calculate the corresponding state of stress components σ_{xx} and σ_{xz} at the point. *(2 marks)*

4 The velocity field in an unsteady moving fluid is given by $\mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$, where $u \equiv u(x, y, z, t)$, $v \equiv v(x, y, z, t)$ and $w \equiv (x, y, z, t)$.

(i) (a) The *divergence* of $\mathbf{V}$ is defined by

$$\nabla\cdot\mathbf{V} \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \lim_{\delta\nu\to 0} \frac{1}{\delta\mathcal{V}} \frac{D(\delta\mathcal{V})}{Dt}.$$

By considering the mass of the moving control volume, $\delta m = \rho\,\delta\mathcal{V}$, derive the equation of continuity (that is, of *mass conservation*) for a compressible fluid and show that this can be rearranged in *conservative* form as

$$\frac{\partial\rho}{\partial t} + \frac{\partial\rho u}{\partial x} + \frac{\partial\rho v}{\partial y} + \frac{\partial\rho w}{\partial z} = 0.$$

(7 marks)

(b) Now, assume an incompressible flow, i.e. $\nabla\cdot\mathbf{V} = 0$, is restricted to the x direction only. Using the notation $\rho_{i,j} = \rho(x_i,\, t_j)$ and $u_{i,j} = u(x_i,\, t_j)$, derive an explicit finite difference expression for $\rho_{i,j+1}$, accurate to order $O(\Delta x^2)$. Set $\Delta x = 0.1$ and $\Delta t = 0.01$.

(3 marks)

(c) State one advantage the implicit method has over the explicit.

(1 mark)

(d) The 3D momentum equation may be written in the form

$$\frac{D\mathbf{V}}{Dt} = -\nabla p + (\nabla^T \cdot [\sigma])^T + \mathbf{F},$$

where $[\sigma]$ is the rate of strain tensor, T indicates the transpose and $\mathbf{F}$ the body forces. The tensor is proportional to the gradient of the velocity:

$$\sigma_{ij} = \mu\left(\frac{\partial V_j}{\partial x_i} + \frac{\partial V_i}{\partial x_j} - \delta_{ij}\frac{2}{3}\frac{\partial V_k}{\partial x_k}\right) + \delta_{ij}\lambda\frac{\partial V_k}{\partial x_k}.$$

Write the tensor in matrix form in terms of u, v and w. ***(4 marks)***

(e) Show that the tensor is traceless, i.e. $\sigma_{ii} = 0$, if $\lambda = 0$. ***(1 mark)***

(ii) $\mathbf{G}$, the curl of a vector field $\mathbf{F}$, may be expressed in index notation as

$$G_i = \varepsilon_{ijk}\frac{\partial F_k}{\partial x_j} \text{ where } i, j, k \text{ may each take any of the values } 1, 2, 3.$$

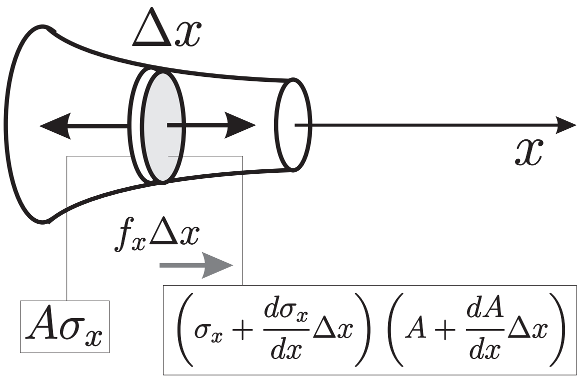
(a) Show that the divergence, $\nabla\cdot\mathbf{G} = \partial G_i/\partial x_i = 0$. ***(3 marks)***

(b) What is G_2? You are given that the components of the Levi-Civita tensor are

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1, \quad \varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1, \quad \text{otherwise} \quad 0.$$

(1 mark)

5 Figure 3 depicts a 1D bar of variable cross-sectional area fixed at one end ($x = 0$) and loaded axially by a known force F_ℓ at its free end ($x = \ell$). The total force acting across the cross-sectional area located at x is denoted by $F(x)$. The body force per unit length is $f(x)$.



$$-A\sigma_x + \left(\sigma_x + \frac{d\sigma_x}{dx}\Delta x\right)\left(A + \frac{dA}{dx}\Delta x\right) + f_x \Delta x = 0$$

Figure 3: Equilibrium forces acting in a one-dimensional bar with variable cross-sectional area

(i) (a) Derive the strong form of the one-dimensional equation of force balance, given the stress-strain relation simplifies to $\sigma_x = E\varepsilon_x$ and the engineering strain $\varepsilon_x = du/dx$. E is the elastic stiffness. ***(5 marks)***

(b) What are the boundary conditions at $x = 0$ and $x = \ell$? ***(2 marks)***

5 (continued)

(ii) Multiplying this strong form by an arbitrary weighting function, w, the weak form can be derived in integral form as

$$\int_0^\ell \frac{dw}{dx} A\, E \frac{du}{dx} dx = w(\ell)F(\ell) - w(0)F(0) + \int_0^\ell w\, f_x\, dx\,.$$

Consider a basic solution domain $0 \leq x \leq \ell$, having only two nodes. The trial solution $u(x) \approx U_1 N_1^1 + U_2 N_2^1 = \mathbf{N}^T\mathbf{U}$ includes constants $\mathbf{U}$, and weight $w(x) = c_1 N_1^1 + c_2 N_2^1 = \mathbf{c}^T\mathbf{N}$ includes arbitrary constants $\mathbf{c}$.

(a) Show that

$$E\left[\int_0^\ell A \frac{d\mathbf{N}}{dx}\frac{d\mathbf{N}}{dx}^T dx\right]\mathbf{U} = \mathbf{N}(\ell)F(\ell) - \mathbf{N}(0)F(0) + \int_0^\ell \mathbf{N}\, f_x dx.$$

(3 marks)

(b) Given a special case where A and f are constant and

$$\begin{pmatrix} N_1^1 \\ N_2^1 \end{pmatrix} = \frac{1}{\ell}\begin{pmatrix} \ell - x \\ x \end{pmatrix}$$

obtain the pair of simultaneous finite element stiffness equations

$$U_1 - U_2 = \frac{\ell}{EA}\left(\frac{\ell\, f_x}{2} - F(0)\right), \quad U_2 - U_1 = \frac{\ell}{EA}\left(\frac{\ell\, f_x}{2} + F(\ell)\right).$$

(6 marks)

(c) Split the domain in two, now with three nodes, and assemble an appropriate set of four finite element stiffness equations.

(2 marks)

(d) From the boundary conditions at $x = 0$ and $x = \ell$ simplify the equations and identify the remaining two unknowns. Note, self-consistency requires $F(0) = \ell\, f_x + F(\ell)$. ***(2 marks)***

End of Question Paper

Formulae Sheet

Notation:

$$U(x_i,\ t_j) \equiv U_{i,j}$$

Forward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t}(x_i,\ t_j) \approx \frac{U_{i,j+1} - U_{i,j}}{\Delta t}$$

Forward difference formula for $\partial U/\partial x$:

$$\frac{\partial U}{\partial x}(x_i,\ t_j) \approx \frac{U_{i+1,j} - U_{i,j}}{\Delta x}$$

Backward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t}(x_i,\ t_j) \approx \frac{U_{i,j} - U_{i,j-1}}{\Delta t}$$

Backward difference formula for $\partial U/\partial x$:

$$\frac{\partial U}{\partial x}(x_i,\ t_j) \approx \frac{U_{i,j} - U_{i-1,j}}{\Delta x}$$

Central difference formula for $\partial U/\partial x$:

$$\frac{\partial U}{\partial x}(x_i,\ t_j) \approx \frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x}$$

Central difference formula for $\partial^2 U/\partial x^2$:

$$\frac{\partial^2 U}{\partial x^2}(x_i,\ t_j) \approx \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{\Delta x^2}$$

Relation between different parameters:

A number of relationships between E, ν, K, λ and μ hold and are summarized in Table 1. μ ($\equiv G$) is the elastic shear modulus, K the elastic bulk modulus, E the elastic stiffness (or Young's Modulus) and ν Poisson's ratio.

	E	ν	K	λ	$\mu \equiv G$
E, ν	-	-	$\frac{E}{3(1-2\nu)}$	$\frac{E\nu}{(1+\nu)(1-2\nu)}$	$\frac{E}{2(1+\nu)}$
E, K	-	$\frac{3K-E}{6K}$	-	$\frac{K(9K-3E)}{9K-E}$	$\frac{3KE}{9K-E}$
K, μ	$\frac{9\mu K}{3K+\mu}$	$\frac{3K-2\mu}{2(3K+\mu)}$	-	$K-\frac{2\mu}{3}$	-

Table 1: The relations between the properties of elastic bodies.