The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2014–15**

Advanced Calculus and Linear Algebra - MAS211 **2.5 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) Give a definition of a basis of a vector space. ***(3 marks)***

(ii) State the Basis Theorem. ***(2 marks)***

(iii) Let $V = \mathrm{Span}(v_1, v_2, v_3)$ and $U = \mathrm{Span}(u_1, u_2, u_3)$ be subspaces of the vector space $\mathbb{R}^4$ over $\mathbb{R}$ where

$$\begin{aligned} v_1 &= (1,0,1,0), \quad & v_2 &= (0,1,1,2), \quad & v_3 &= (0,2,2,4), \\ u_1 &= (0,1,0,1), \quad & u_2 &= (1,-1,2,0), \quad & u_3 &= (1,0,2,1). \end{aligned}$$

Find bases for the vector spaces V, U and $V \cap U$. ***(7 marks)***

(iv) Find the complete solution to the system of linear equations

$$\begin{cases} x_1 + x_2 - 2x_3 - x_4 = 3 \\ 2x_1 + 3x_2 - 5x_3 - 4x_4 = 8 \\ 3x_1 + 4x_2 - 7x_3 - 5x_4 = 11. \end{cases}$$

(6 marks)

(v) Find the determinant

$$\begin{vmatrix} 1+x & 1 & 1 & 1 & \dots & 1 \\ 2 & 2+x & 2 & 2 & \dots & 2 \\ 3 & 3 & 3+x & 3 & \dots & 3 \\ & & \vdots & & & \\ n & n & n & n & \dots & n+x \end{vmatrix}.$$

(7 marks)

2 Let $\mathbb{R}[x]_{\leq 2} = \{a_0 + a_1 x + a_2 x^2 \mid a_i \in \mathbb{R}\}$, $\partial = \dfrac{d}{dx}$ and

$$A = 1 + \partial + 2\partial^2 + 7\partial^3 : \mathbb{R}[x]_{\leq 2} \to \mathbb{R}[x]_{\leq 2}, \quad p \mapsto Ap.$$

(i) Find the matrices $M(A)$ and $M(B)$ of the linear maps A and

$$B := 1 + 2A + 2A^2$$

with respect to the bases $e = (1, x, x^2)$ and $e' = (1 + x, 1 - x, -1 + x + x^2)$ of the vector space $\mathbb{R}[x]_{\leq 2}$ (i.e. $M_{e,e}(A)$, $M_{e,e}(B)$, $M_{e',e'}(A)$ and $M_{e',e'}(B)$). ***(12 marks)***

(ii) Show that the linear maps A and B are invertible and find A^{-1}. ***(8 marks)***

(iii) Find the complete solution $f = a + bx + cx^2 \in \mathbb{R}[x]_{\leq 2}$ of the differential equation

$$7\frac{d^3 f}{dx^3} + 2\frac{d^2 f}{dx^2} + \frac{df}{dx} + f = 1 + x + x^2.$$

(5 marks)

3 (i) State Green's Theorem. Using Green's Theorem evaluate

$$\int_C (3xy + 2xe^{x^2+y^3})dx + (\frac{8}{3}x^3 + 3y^2 e^{x^2+y^3})dy,$$

where C is the triangular path with vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$ described in the anticlockwise direction. ***(13 marks)***

(ii) Let T be the closed curve consisting of the arc of the curve $y = x^2$ from the origin to $(1, 1)$ followed by the arc of the curve $x = y^2$ from $(1, 1)$ back to the origin. Evaluate

$$\int_T 4xy dx + (6x + 6y)dy.$$

(12 marks)

4 (i) Let f be the periodic function with period 2π such that $f(x) = e^x$ for $-\pi < x < \pi$.

(a) Sketch the graph of f. Is the function f odd, even or neither? Justify your answer. ***(4 marks)***

(b) Calculate all the coefficients in the Fourier series for f. ***(8 marks)***

(ii) Find the divergence $\mathrm{div}(f)$ where $f = e^{x^2+y^2+z^2}$. ***(2 marks)***

(iii) State the Divergence Theorem. ***(3 marks)***

(iv) Let ∂G be the boundary of the region

$$G = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \le z \le 1, \;\; 0 \le y \le z, \;\; 0 \le x \le y + 2z\}.$$

By using the Divergence Theorem (or otherwise) find the surface integral

$$\int\int_{\partial G} (\mathbf{f}, \mathbf{n}) dS$$

where $\mathbf{f} = (x^2 + e^{\sin(y)}, y^2 + y\cos(yz), 2zx - z\cos(yz))$.

(8 marks)

End of Question Paper