Data provided: formula sheet

MAS241

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2014–15**

Mathematics II (Electrical) **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

Please leave this exam paper on your desk
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Registration number from U-Card (9 digits)
to be completed by student

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1 (i) Consider the differential equation

$$y''(t) + 4y(t) = 1$$

subject to the initial conditions $y(0) = 1$ and $y'(0) = 0$.

(a) Show that the Laplace transform of $y(t)$ is given by

$$Y(s) = \frac{s^2+1}{s(s^2+4)}.$$

(8 marks)

(b) Use the inverse Laplace transform to find $y(t)$ at time $t > 0$.
(6 marks)

(ii) Calculate

$$\int_{-\infty}^{\infty} \delta(t-\pi)\sin(\tfrac{t}{3})e^{(t-\pi)j}dt.$$

(3 marks)

(iii) Find the Fourier transform of the function $f(t) = \text{sinc}(t)$. **(3 marks)**

2 (i) Consider the periodic function $f(t)$ with fundamental period 2 defined by

$$f(t) := \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } 1 < x < 2 \end{cases}$$

Find the Fourier series of $f(t)$. **(14 marks)**

(ii) The Fourier cosine series of the function $f(x) = x^2$ on $[0,1]$ is given by

$$S[\bar{f}](x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{\pi^2 n^2}\cos(n\pi x).$$

(a) Sketch the graph of $S[\bar{f}](x)$ over the interval $[-2,2]$. **(4 marks)**

(b) By considering $S[\bar{f}](x)$, find an exact value for

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

(2 marks)

3 (i) Let $f(x, y) = \sin(x^2 + y^3)$. By calculating both $f_{xy}(x, y)$ and $f_{yx}(x, y)$ directly, verify that $f_{xy}(x, y) = f_{yx}(x, y)$. **(5 marks)**

(ii) Find and classify *all* the critical points of the function

$$f(x, y) = 2x^3 - 6xy + 3y^2.$$

(10 marks)

(iii) Let $f(x, y) = (x + 2y)^2$, $x(u, v) = e^{u^2+v^2}$ and $y(u, v) = e^{2uv}$. Calculate the partial derivative $\dfrac{\partial f}{\partial u}$. **(5 marks)**

4 (i) Let $T \subset \mathbb{R}^2$ be the region bounded by the lines $y = x$, $y = -x$, and $x = \sqrt{\pi}$, and let $f(x, y) = x^2 \cos(xy)$. Find

$$\iint_T f(x, y)dA$$

(10 marks)

(ii) Express the point $(x, y, z) = (1, 1, \sqrt{2})$ in both spherical polar and cylindrical polar coordinates. **(2 marks)**

(iii) The region

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$$

has a density of

$$d(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

at (x, y, z). Find the mass of B. **(8 marks)**

5 (i) Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by

$$\mathbf{F} = \left(\frac{x}{2}, \frac{y}{2}, -\frac{x^2 + y^2}{z} \right).$$

(a) Calculate **div F**. **(4 marks)**

(b) Calculate **curl F**. **(4 marks)**

(ii) Let $f(x, y) = \sin(\frac{\pi}{4}x)\cos(\frac{\pi}{4}y)$.

(a) In which direction is the graph of $f(x, y)$ most rapidly decreasing at the point $(1, 1)$? **(4 marks)**

(b) Calculate the directional derivative of $f(x, y)$ at $(-1, 1)$ in the direction of $\mathbf{v} = (1, -1)$. **(4 marks)**

(iii) Sketch the vector field $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\mathbf{F} = (-y, x).$$

(4 marks)

End of Question Paper